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Work-In-Process inventory and production capacity

Sung Ho Chung
Iowa State University

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Sung Ho Chung

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I. INTRODUCTION

Innumerable papers and books have been written about inventory control. However, inventory control is one of the areas where a wide gap exists between theories and practices in spite of the abundance of literature. Although they do not compose an exhaustive list of the reasons for the gap, the following seem to be part of it.

First, the situation and the environment in which a company operates are unique. The way problems are handled has been established during a long time period. Consequently, no models are general enough to fit for every company.

Second, inventory models are generally too theoretical for a layman to understand. In addition, it is difficult to collect data for the variables and parameters defined in the models.

Third, in most of the inventory models, it is assumed that anything happening in the inventory theory does not affect the rest of a company. However, this independence is far from reality.

Fourth, practitioners consider inventory as a necessary evil. They are looking for a model which could given them a magic number to minimize the harm done by the evil. They seldom think that inventory is an asset, like production facilities.

Fifth, inventory models do not suggest the way to implement them. Also no idea is given about the costs associated with implementation.

Although there is a lot of work yet to be done to narrow the wide gap, more attention and effort is currently given toward developing workable models and techiques. The communication and cooperation between theoreticians and practitioners seems to be increasing.

In the midst of the struggle for narrowing the gap, Work-In-Process inventory has been given the least consideration. There are two conceivable reasons for this. First, the dollar value of the WIP inventory is relatively small compared to raw materials and finished products. (McRoberts and Chung, 1975) Second, it is difficult to analyze the WIP inventory because of its complex relation with production scheduling.

Recently the author had a chance to correspond with an experienced consultant in the field of production planning and control. He said, "No work has been done on an estimate of how much capital might be removed from work-in-process in industries in this country. My own feeling is that most companies could reduce work-in-process levels 50% and find only good would result. Think of the capital this would free up! Technically there is little difficulty in doing this--the big problem occurs because people's intuition tells them they need large cushions of work in the plant to run economically." (Plossl, 1975)

The objective of this research is to shed some light on one fundamental question, "What is the optimum level of the WIP inventory for a production system?" It is very difficult to answer this question because the optimum level depends on various attributes of the system. Among these are the goals of production planning and control, which are often contradicting each other, production resources, financial resources, type of production, type of products, demand patterns, etc. Unless a set of optimization criteria is specified and a reasonably simple production system is hypothesized, answering the question seems to be an insurmountable job. Hence the approach taken in this research is to examine the WIP inventory of hypothesized production systems where the criterion of optimization is cost minimization. The optimum level of the WIP inventory of the systems will be determined by an optimum solution which minimizes the cost considered.

Two types of production system are studied in this research. Both types produce multiple parts with multiple machine centers. Each machine center is composed of one or multiple identical machines. The first type is a deterministic production system where various parameters of the system, such as demand rates and production rates, are known constants. Its production scheduling is simplified a great deal by assuming that each machine center produces incoming parts cyclically with integer number of cycles per year. The second type is a stochastic production system where its parameters are random variables having known probability distributions. Its production scheduling is simplified by assuming that each part is produced by a lot of which size is predetermined and the queue discipline at each machine center is first-come-first-served.

In the deterministic system the prime objective of the study is to derive a functional relationship between the average level of the WIP inventory and the production cycles of individual machine centers.

The second objective is to find an optimum set of production cycles which minimize the production cost. Initially the production cost includes only setup cost and WIP inventory holding cost. As an extension of the study, the production capacity of each machine center, which is defined as the total available machine hours per year at each center, is taken into account in finding an optimum solution. To be more specific, the system incurs a fixed amount of cost for carrying each machine at each machine center. This cost is included in the production cost to be minimized. The capacity consideration is introduced into the study in two different ways. One is to assume that the production capacity is non-deteriorating over time and there is no machine replacement. The other is to assume that the production capacity is deteriorating over time and machine replacement is allowed. In the latter case, technological improvement is also considered. Although the decisions on capacity and machine replacement seem to be appropriate factors to be considered in inventory models, no attention has been given to these in the literature. Likewise, no work has been done for considering inventory in machine replacement models. One objective of the study is, accordingly, to combine the decision on inventory and that on machine replacement effectively by the assumption of deteriorating capacity.

The machine replacement model is studied for the case where there is no budget constraint and where there is some type of budget constraint.

In the stochastic system the prime objective is to derive empirical functional relations between the mean and variance of the production lead time of each part, and the number of production orders to the system and the service rates of machine centers. Two hypothesized production systems are simulated and the mean and variance of each part are observed at various levels of production orders and service rates. The data collected for mean and variance are regressed and empirical functional relations are obtained.

The second objective is to find an optimum trade-off point in terms of cost among the WIP inventory, the number of production orders and the service rates of machine centers. The WIP inventory of the system is divided into two groups; one of them is the WIP in production floor and the other is the WIP in the Finished Piece Parts Storage. There is a functional relationship between the former and the mean of the production lead time of each part. There is also a functional relationship between the latter and the mean and variance of the production lead time of each part. By identifying these relations and using the empirical relations obtained from the simulation and regression analysis, an optimum trade-off point is located for the hypothesized systems.

The main body of this dissertation is Chapter III and Chapter IV.

Chapter III deals with the deterministic system and it begins with a method to calculate the WIP inventory between two adjacent machine centers. Although several restrictive assumptions are made at the beginning, some of them are relaxed at later sections. An optimization scheme is discussed in Section B for the mathematical model developed in Section A. Section C and Section D deal with non-deteriorating production capacity and machine replacement policy, respectively,

Chapter IV is concerned with the stochastic system. In Section A the functional relations between the WIP inventory and production lead time are discussed. Also a brief discussion on current queueing theory and its applicability to the stochastic system is presented. The last part of Section A is devoted to solving a single server queueing system with heterogeneous customers. The simulation study and regression analysis are presented in Section B.

II. LITERATURE REVIEW

An early paper on WIP inventory by Simpson (1958) looked at manufacturing operations as chains of operations separated by inventories. By assuming that the system of manufacturing operations was the base-stock system, he solved for the optimum level of inventories which minimized a linear inventory holding cost. This paper also addressed the question of what points in the manufacturing operation should be inventory stocking points and what points should not. The demand for the final product was assumed to be a random variable having known mean and variance but unknown distribution. Clark and Scarf (1960) studied a system which is similar to Simpson's. They considered the problem of determining optimal purchasing quantities at individual inventory stocking points. The cost to be minimized included purchasing cost, linear holding cost, and linear shortage cost. By making several plausible assumptions, the optimum solution was obtained by techniques which had been used for the computation of optimal policies at a system having a single inventory stocking point. The demand for the final product was assumed to be a random variable having a known distribution function.

Taha and Skeith (1970) developed a model for a single-product multistage production system with deterministic demand, where the product moves between the stages in a serial fashion. The production rate at each stage was not instantaneous and there were time lags between stages. The decision variables, the batch size of the finished product, production quantity per run at each stage, and the shortage quantity of the finished product were determined by minimizing the total cost per unit time which included linear holding cost, linear shortage cost, and fixed setup cost. One key assumption of the model was that the production quantity per run at stage i is an integer multiple of that at stage $i+1$.

Crowston et al. (1973) considered the problem of economic lot size determination in multi-stage assembly systems where each facility had many predecessors but only a single successor. Assumptions included constant continuous final product demand, instantaneous production, and an infinite planning horizon. Under the constraint that lot sizes remained time invariant, they proved that the optimal lot size at each facility was an integer multiple of that at the successor facility. They solved for optimum lot sizes by N stage dynamic programming with some appropriate computational refinements. Schwarz and Schrage (1975) examined a system similar to the one considered by Crowston et al. Their objective was to select ordering policies which minimized (or nearly minimized) average system cost per unit time over an infinite planning horizon when the customer demand rate was constant. The system cost included fixed setup cost and linear holding cost.

The above five papers dealt with a multi-stage production/ inventory system. However, none of them handled the problem of machine scheduling properly. Some ignored machine scheduling and some made assumptions which virtually uncoupled the inventory and scheduling problems.

WIP inventory has been studied extensively in a production line environment. The main interest of this line of study is to determine optimum storage capacities at individual inventory stocking points.

Koenigsberg (1959) reviewed the basic problems associated with the efficient operation of production and assembly lines, and evaluated the effectiveness of internal storage. He discussed three basic approaches to the problem and made a three-way comparison among them.

Anderson (1968) developed cost models for several types of production lines. Based on data from a production line simulation, regression equations were developed for estimating the average delay and average in-process inventory. By using the regression equations and estimates of the appropriate costs, the total cost for each model is expressed as a function of the number of stages and the storage capacity. Special consideration was given to establishing the minimum cost storage capacities during the transient or start-up phase of the production run. Shamma et al. (1973) did a type of study similar to Anderson's.

Buzacott (1971) discussed the effects of the number, location and capacity of inventory bank on flow-line production system. Some quantitative results were presented.

A few papers treated the WIP inventory from a practical point of view. Wight (1970) discussed the way to control production lead time and WIP inventory. He considered backlogs at individual machine centers the fundamental cause of longer lead time and high level of WIP inventory. He claimed that the only way to control lead time was to control backlogs. He listed three causes of large backlogs and discussed how these causes should have been handled.

Plossl (1971) discussed the problem of determining proper level of various types of inventory. He classified inventory by its function and discussed the roles, benefits, problems, and relations with other factors and parameters of a production system for each class.

Plossl and Wight (1973) reviewed and examined various aspects of production planning and control. The discussion covered existing techniques, associated problems, proper ways to handle the problems and principles for each aspect with particular emphasis on lead time control.

Bell (1973) criticized the implicit assumptions underlying the EOQ-ROP inventory model. He put inventory in a new perspective and discussed the relation between inventory and the remaining sector of a company, and the relation between inventory and customers.

It is commonly assumed in inventory theory that procurement lead time is constant. However, this assumption is not representative of most situations while variable lead time presents some inherent theoretical difficulties. Bramson (1962) did a survey of the literature on this subject. He presented different approaches to the problem for each class of inventory model. Clark and Rowe (1960) came up with an approximate relation among order quantity, reorder point and the fraction of stock out for a general lead time demand. Ekey et al. (1961) calculated the probability of a shortage during reorder cycle in terms of lead time distribution and demand distribution.

Gross and Soriano (1969) examined the effect of reducing lead time on inventory levels via simulating a military overseas supply system. They observed the effect for each different combination of lead time distribution and demand distribution.

Silver (1970) suggested a modified formula for calculating customer service which was measured by the fraction of the time during which demand is satisfied without backorders. Numerical results were provided.

Burgin (1972) developed an exact expression for protection and potential lost sales for a continuous review inventory model in which the demand is normally distributed and the lead time gamma distributed.

Danish (1972) studied the problem of calculating the reorder point for a continuous review inventory model. The reorder point was calculated for each different combination of lead time distribution and demand distribution.

Since the stochastic production system of Chapter IV is a queueing network, the literature of queueing theory was reviewed. The two papers by Jackson (1957 and 1963) discussed the stationary solution of jobshop-like queueing system. Ancker and Gafarian (1961) and Kotiah and Slater (1973) studied a queueing system with heterogeneous customers. Rosenshine (1975) and Disney (1975) reviewed and summarized the theory of queue. All these papers written on queueing theory are discussed in greater detail in later sections.

Shore (1975) considered machine replacement decisions under capital budgeting constraints. His model was an extension of Terborgh's model (1949). He developed a formula to compute the net benefit to be realized by replacement. Using this formula, a zero-one integer programming model was developed in which the objective function was the total net benefit and constraints were yearly budget. The formula of the net benefit was a very complicated one. However, it is not necessary to use the formula to compute the net benefit since the same result can be acquired from the adverse minimum defined by Terborgh.

III. DETERMINISTIC CASE

A. Development of Work-In-Process Inventory Calculation

1. Introduction

Section A deals with a method which makes it possible to calculate WIP inventory holding cost systematically. Although many restrictive assumptions are made at the beginning, some of them are relaxed in the later part of this section. The method is the corner stone of the mathematical models to be developed in later sections, which in turn will be the basis for determining optimum production and capacity decisions.

While the main purpose of developing the method is to calculate WIP inventory holding cost in a pretty general case where there are N parts and M machine centers, the case of 1 part and M machine centers is dealt with at the beginning for the sake of simplicity.

The part is processed through a series of M machine centers. Figure 3.1 depicts the production system of 1 part and M machine centers.

Each machine center is composed of several identical machines. They are identical in terms of production speed for a given part, setup cost for a given part, available machine hours per year, maintenance cost, etc. The production capacity of a given machine center is measured by the total available machine hours per year.

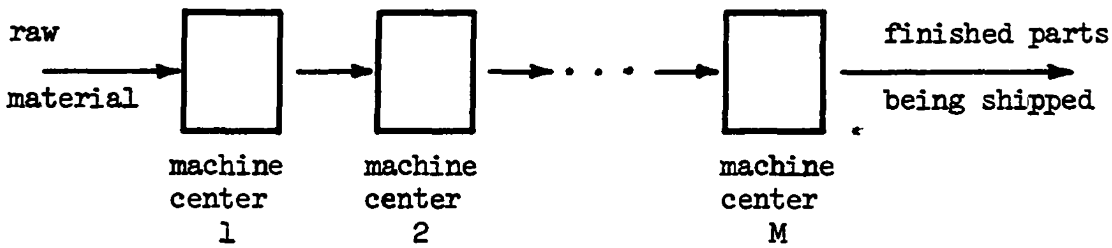

Figure 3.1. Production system of 1 part and M machine centers

Before proceeding, the following assumptions and definition of symbols are made.

Assumptions:

(1) Each machine center produces the part cyclically.

(2) Each machine center has one unique number of cycles per year. The number of cycles per year is integer.

(3) The maximum number of cycles a machine center can have is given.

(4) The demand rate of the part is a known constant.

(5) The production rate at each machine center is a known constant and it is bigger than the known demand rate.

(6) The part is infinitely divisible.

(7) The processed parts at one machine center will be fed into a next machine center continuously.

(8) The moving time of processed parts from each machine center to a next one is a known constant.

(9) Back order is not allowed. So there is no material shortage at each machine center.

(10) There are no defectives.

(11) Back tracking is not allowed.

(12) The setup cost per cycle at each machine center is a known constant.

(13) The dollar value per unit of the part which has been processed at a machine center is a known constant.

(14) The total available machine hours per year of each machine center is infinite.

(15) After the completion of the operation at machine center M, the part will be delivered to a shipping area. The rate of shipment is continuous and equal to the known demand rate.

(16) At a given machine center the part cannot be worked on by more than one machine simultaneously.

(17) No interruption is allowed during a production period of a part.

Assumption 2 regarding the number of cycles per year is made for two reasons. The first one is managerial convenience. The second is the fact that the integer assumption makes it possible to find an optimum solution by using dynamic programming and branch and bound techniques.

The amount of inventory in transfer from one machine center to another is constant because of assumption 8. So the portion of total WIP inventory holding cost due to this amount is constant. Since a constant term does not affect an optimum decision, it is not necessary to consider the amount in calculating the WIP inventory holding cost which is relevant to an optimum decision. Consequently, this amount will be ignored in the development of WIP inventory calculation.

The dollar value per unit mentioned in assumption 13 needs to be refined. Generally it is very difficult to measure the dollar value per unit of a part which is in an intermediate production stage. It is the sum of all relevant costs associated with the part up to that particular production stage. Although it may be relatively easy to keep track of the direct labor costs and material costs involved, finding the portion of the total overhead cost of a manufacturing company associated with that particular part at a certain production stage is certainly a complex task. However, the assumption is made for the development of an easy and useful way of calculating WIP inventory. Note that the dollar value includes direct labor costs, material costs and all other pertinent costs which occur due to the production of the particular part up to a particular production stage.

The assumptions 7, 11, and 14 will be relaxed at later sections.

Definition of symbols:

D: Demand rate (units/yr)

N_i: Number of cycles/yr at machine center i ($i = 1,2,\ldots, M$)

$\bar{N}_i$: The maximum number of cycles per year at machine center i

V_i: \$/unit of the part which has been processed at machine center i

S_i: Setup cost/cycle at machine center i

P_i: Production rate at machine center i

I: Yearly inventory carrying charge (\$/\$-year)

Consider machine center i which will produce D/N_i units per each cycle. Each completed part will be fed into machine center $i + 1$ continuously. The problem is to find the WIP inventory between machine centers i and $i + 1$. The amount of the WIP inventory in transfer, referred to as pipe line inventory, is constant due to assumption 8. Ignoring this pipe line inventory is equivalent to ignoring the moving time. The entire system could be described as if the parts are moved instantaneously from one machine center to another by assumption 8. Consequently, it should be noted that the WIP inventory between machine centers i and $i + 1$ represents only the amount which is not involved in transfer.

In finding the WIP inventory between machine center i and $i + 1$, the following diagram is found to be very useful. It is called cumulative production-demand diagram. This diagram is shown in Figure 3.2.

In order to simplify the situation, one additional assumption is made, i.e., the production rate at machine center i and the rate at machine center $i + 1$ are the same. For this simple situation, Figure 3.3 shows the cumulative production-demand diagram of machine center i and that of machine center $i + 1$. The assumption of equal production rates at machine centers i and $i + 1$ will be relaxed later.

In Figure 3.3, t_i is the moving time from machine center i to machine center $i + 1$. The saw tooth line AB represents the actual cumulative production at machine center i. The saw tooth line CD

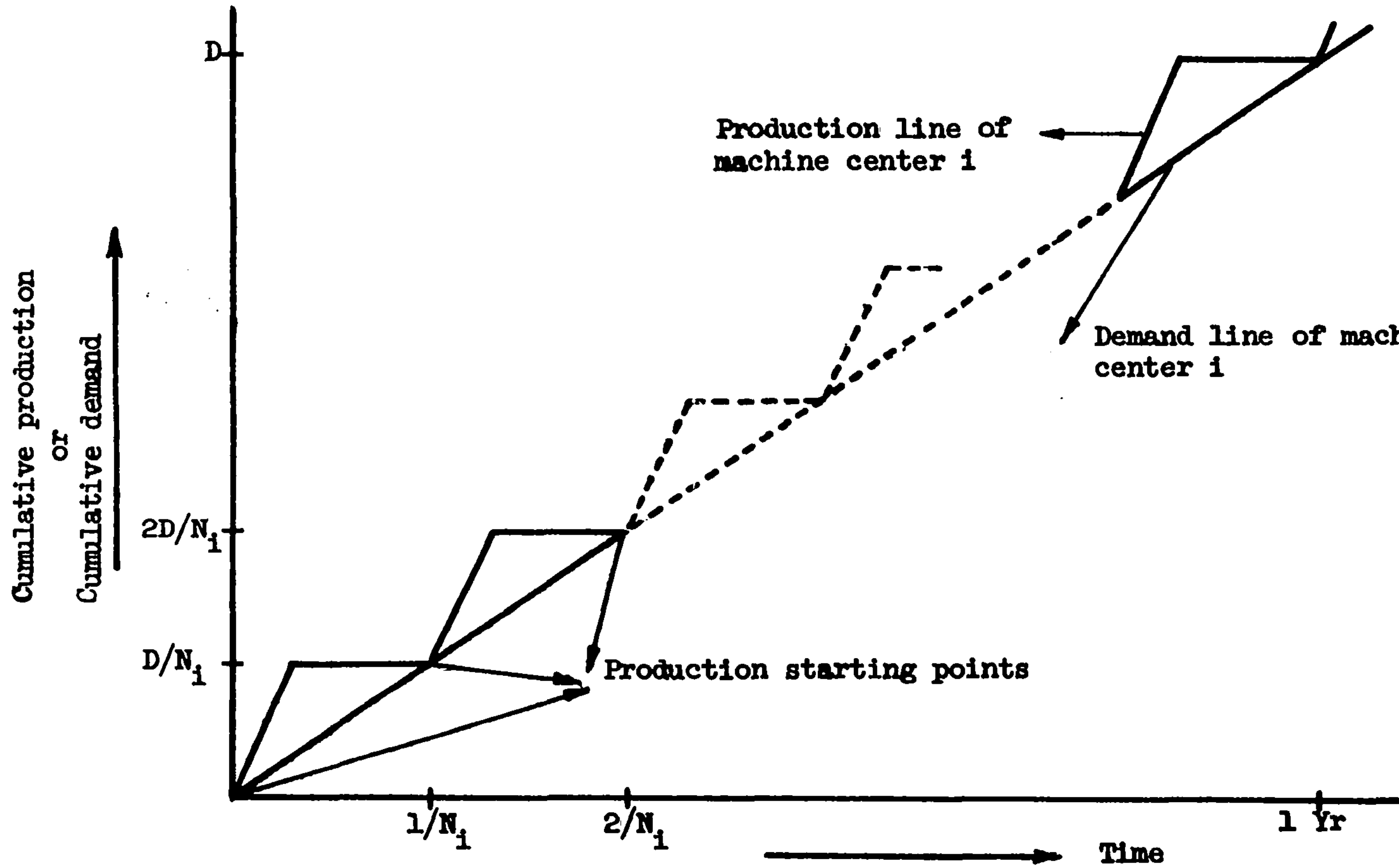

Figure 3.2. Cumulative production-demand diagram

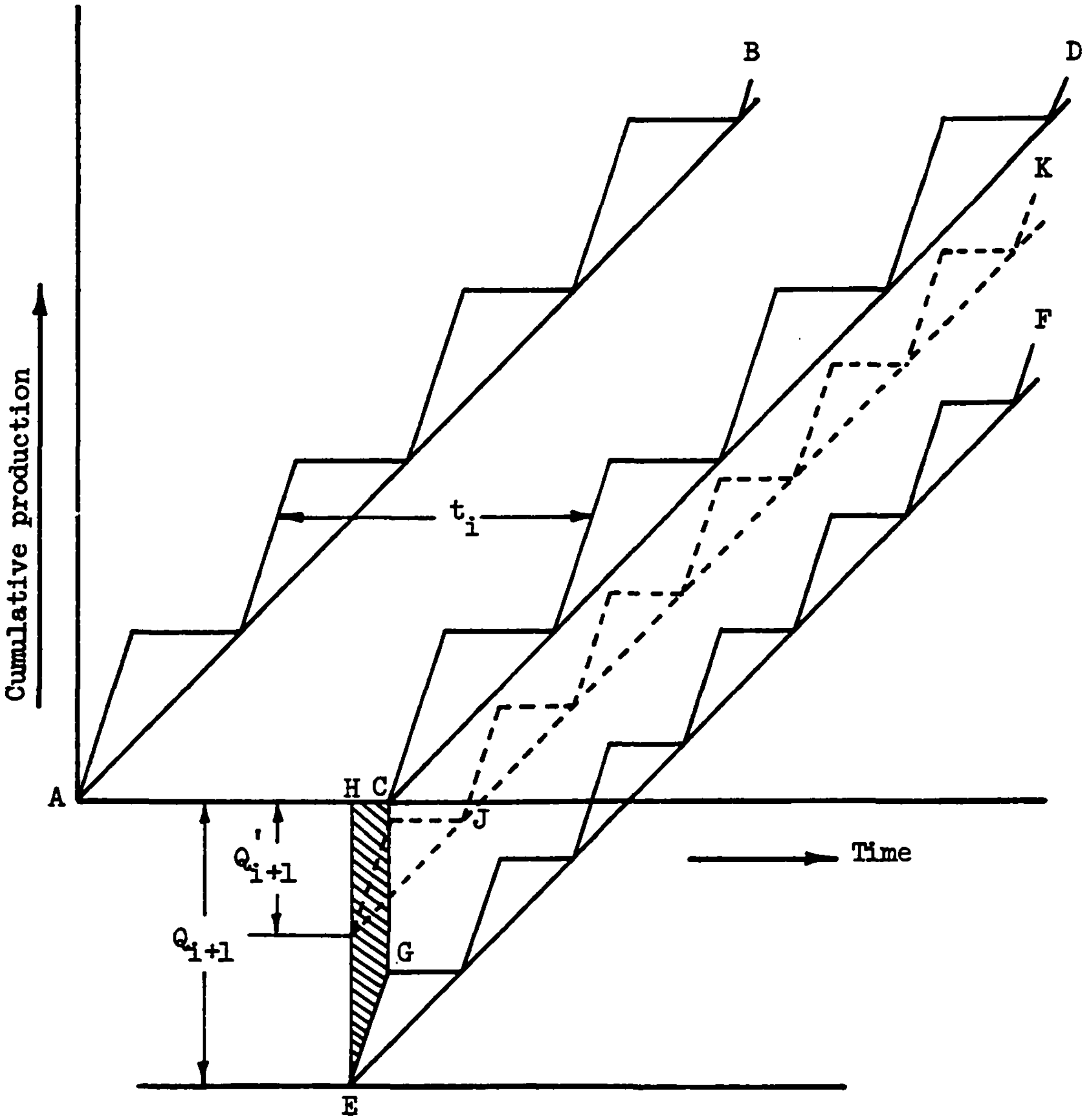

Figure 3.3. Cumulative production-demand diagrams of machine centers i and i+1

represents the cumulation of the parts which have arrived at machine center $i + 1$. AB is called the actual production line of machine center i and CD is called the available production line at machine center $i + 1$. EF is the actual production line of machine center $i + 1$.

The amount of inventory tied up with the transit from machine center i to machine $i + 1$ in terms of unit-years is the area between AB and CD. Since the area is a constant for a given t_i and also independent of N_i, it will not be considered for the WIP inventory calculation between the two machine centers as stated before.

AH is the time lag between the very first production starting points of machine centers i and $i + 1$. Q_{i+1} represents the beginning inventory at machine center $i + 1$. It is assumed that this beginning inventory could be acquired by some means such as purchasing or subcontracting. With the beginning inventory Q_{i+1}, the WIP inventory between machine centers i and $i + 1$ in terms of unit-year is the area between the two saw tooth lines CD and GF plus the shaded areas, EHCG. Since the time horizon is infinite, the area EHCG can be ignored in calculating the average WIP inventory per year. Accordingly the WIP inventory between the two machine centers is the area between CD, which is the available production line at machine center $i + 1$, and GF which is the actual production line of machine center $i + 1$ from which the very beginning section, EG, has been cut off. It should be noted that when the time horizon is finite, ignoring EHCG may not be justified.

In Figure 3.3 it is possible that machine center $i + 1$ starts its production at point H but with beginning inventory much less than Q_{i+1}. The dotted line is the realization of EF by upward shifting. In this case the beginning inventory required is Q'_{i+1}. By the upward shift, WIP inventory between machine centers i and $i + 1$ has been reduced substantially. Before the shift, WIP is represented by the area between GF and CD. After the shift, it is reduced to the area between JK and CD. The WIP inventory between machine centers i and $i + 1$ can be reduced by this manner as long as CD covers GF completely from above.

Figure 3.4 is the reproduction of Figure 3.3 except that the initial production starting point of machine center $i + 1$ occurs much later compared with Figure 3.3.

Q_{i+1} is the beginning inventory at machine center $i + 1$. For this case the WIP is the area between the saw tooth lines GD and EF while ignoring the area LCGE.

It is possible that machine center $i + 1$ starts its production at point H but with zero beginning inventory. The dotted line HI is the realization of EH by upward shifting. By this upward shift the WIP inventory is reduced to the area between GD and HI. It is also possible to reduce the WIP inventory to the area between GD and JK by taking away the incoming parts from machine center i as much as Q'_{i+1}. For example, Q'_{i+1} can be sold out. The WIP inventory can be reduced by this manner as long as the saw tooth line GD covers EF completely from above.

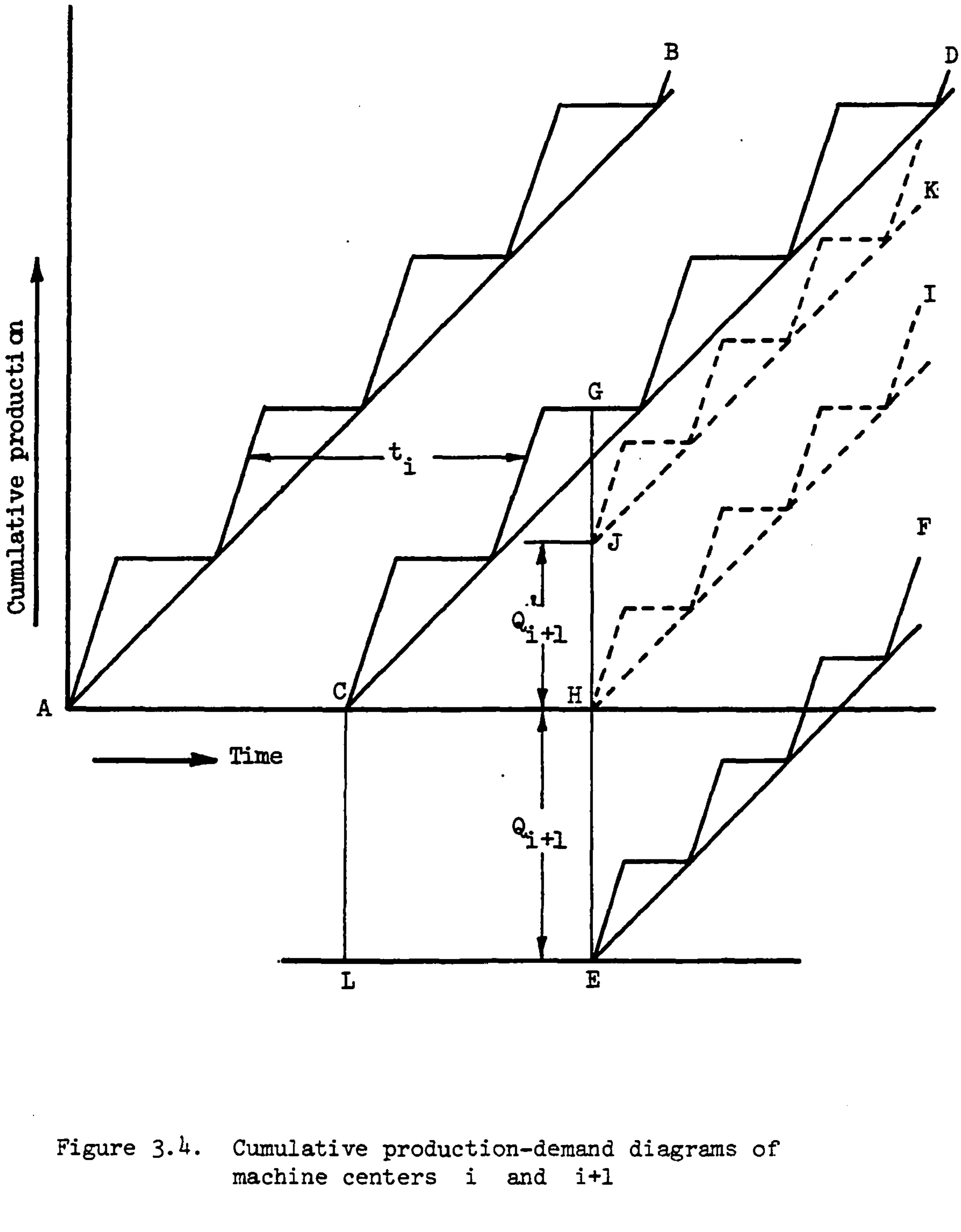

Figure 3.4. Cumulative production-demand diagrams of machine centers i and i+1

From Figures 3.3 and 3.4 it is obvious that the WIP inventory between machine centers i and $i + 1$ is the area between the two saw tooth lines, one of which is the available production line at machine center $i + 1$ and the other is the actual production line of machine center $i + 1$. It seems logical to assume that the manager of this production system will try to reduce the area between the two saw tooth lines as much as possible by acquiring a minimum beginning inventory or taking away a maximum amount of incoming parts from machine center i depending on the relative locations of A and H. The actual quantities of those minimum and maximum also depend on the locations of A and H.

Since the planning horizon is assumed to be infinite, the beginning minimum inventory to be acquired or the maximum amount of incoming parts to be set aside will have negligible effect on the average yearly WIP inventory between the two machine centers. The problem is reduced to calculating the area between the two saw tooth lines where the area has been reduced as much as possible by acquiring a minimum beginning inventory or taking away a maximum amount of incoming parts from machine center i.

Figure 3.5 shows the two saw tooth lines when the area between them has been reduced as much as possible. In Figure 3.5 the production inventory of machine center i is defined to be the area surrounded by the available production line and the demand line of machine center i. The installation inventory of machine center i is defined to be the area surrounded by the demand line of machine center i and that of machine center $i + 1$.

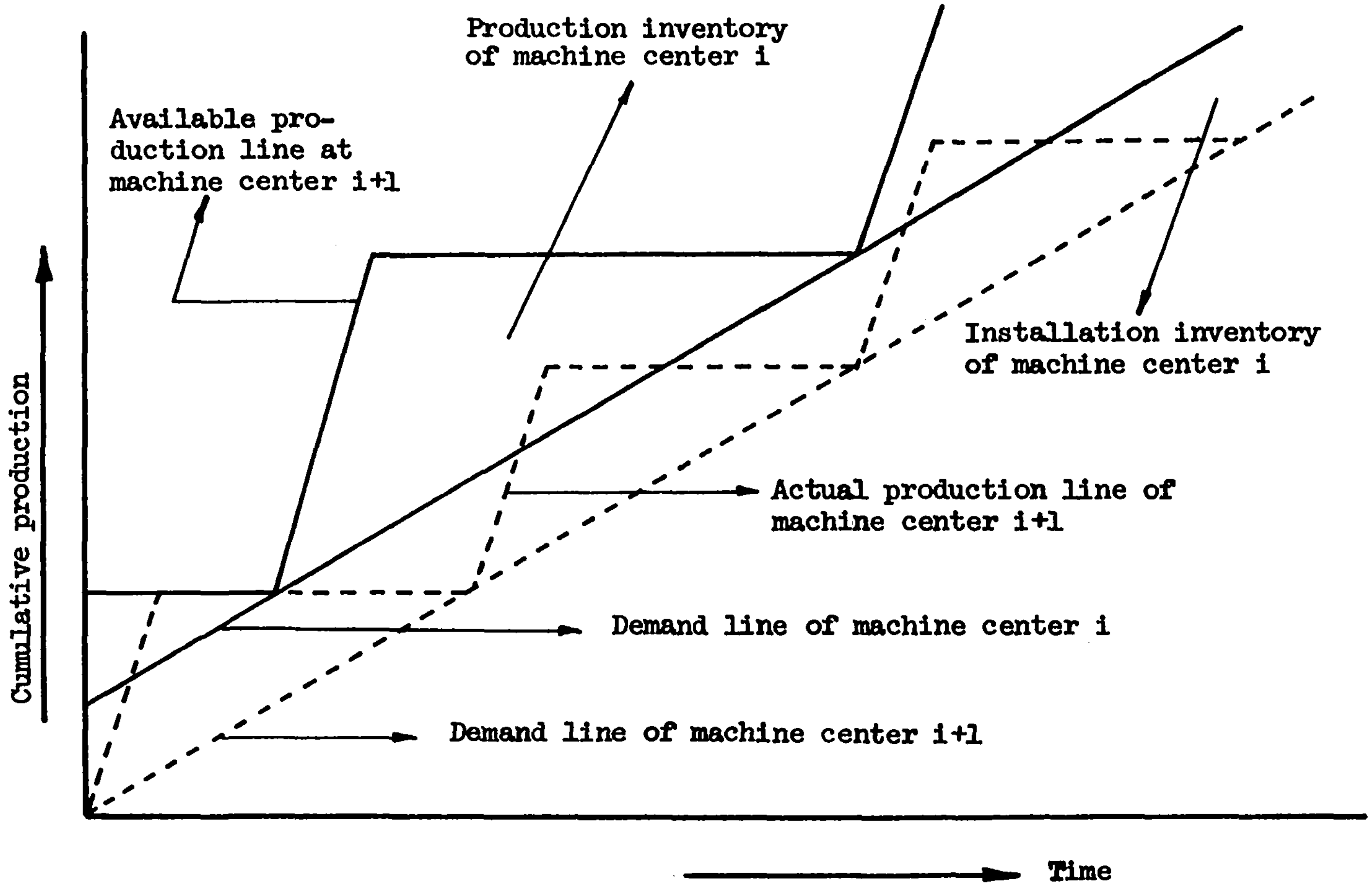

Figure 3.5. The production inventory and installation inventory of machine center i

The WIP inventory between machine center i and i + 1 is the area surrounded by the available production line and the actual production line. Figure 3.6 shows this inventory.

Note that the production inventory, the installation inventory and the WIP inventory in Figures 3.5 and 3.6 are all represented in terms of unit-years.

It is obvious from Figures 3.5 and 3.6 that the WIP inventory between machine centers i and i + 1 is the sum of the production inventory and installation inventory of machine center i less the production inventory of machine center i + 1. In an equation form,

(WIP inventory between machine centers i and i + 1)
= (production inventory of machine center i)
+ (installation inventory of machine center i)
- (production inventory of machine center i + 1) . (3.1)

If the production inventory and the installation inventory of each machine center could be expressed in a simple equation in terms of known constants, it would be possible to calculate the amount of the WIP inventory between each pair of machine centers. Equation 3.1 plays the key role in calculating the WIP inventory in later sections.

2. The production inventory of machine center i

Consider the production inventory of machine center i during one cycle represented by the triangle ABC in Figure 3.7. In Figure 3.7, Δ BCH and Δ EFG are the same size and Δ ABH and Δ AEG are the same size. Consequently, Δ ABC and Δ AEF are the same size.

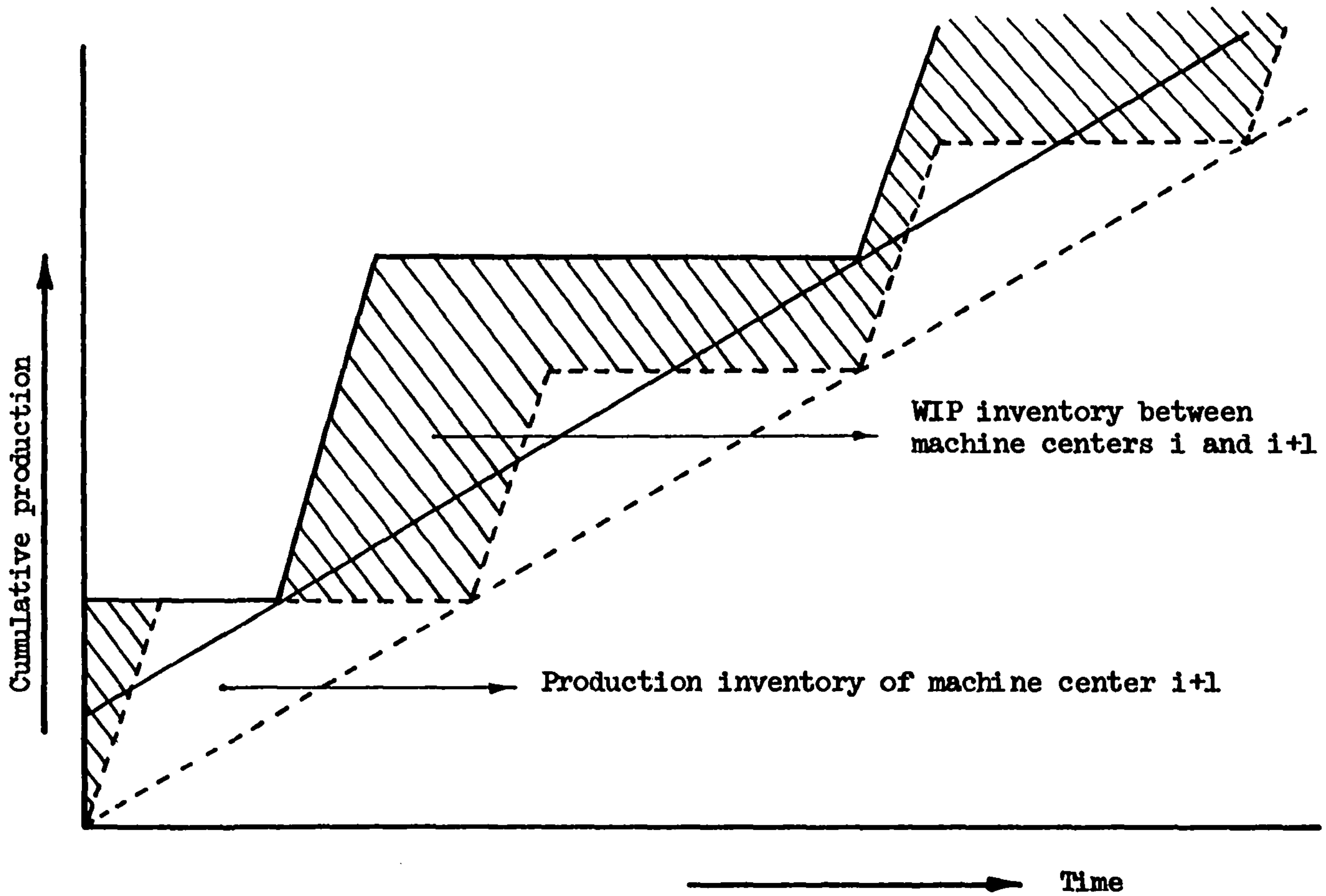

Figure 3.6. WIP inventory between machine center i and machine center i+1

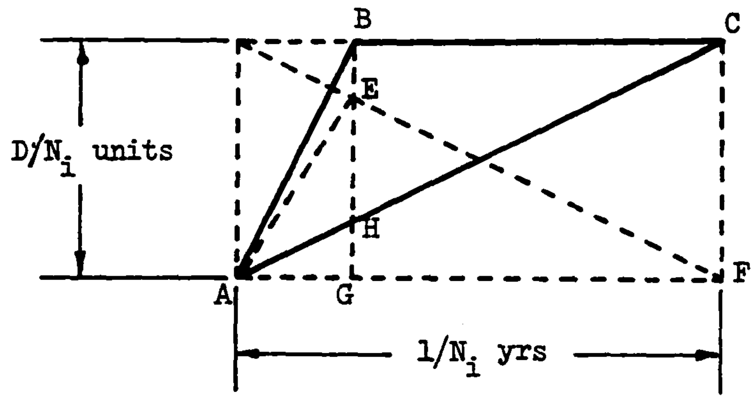

Figure 3.7. The production inventory of machine center i during one cycle

Since EG = $\frac{D}{N_i}(1 - \frac{D}{P_i})$, the area of Δ ABC or Δ AEF = $\frac{1}{2} \cdot \frac{1}{N_i} \cdot \frac{D}{N_i}(1-\frac{D}{P_i})$. The production inventory during one year is composed of N_i of Δ ABC's. So the production inventory of machine center i per year can be expressed as

$$\frac{1}{2} \cdot \frac{1}{N_i} \cdot \frac{D}{N_i} \cdot (1 - \frac{D}{P_i}) \cdot N_i = \frac{1}{2} \cdot \frac{D}{N_i} \cdot (1 - \frac{D}{P_i}) .$$

3. The installation inventory of machine center i

Consider the installation inventory of machine center i per year in Figure 3.5. Its quantity in terms of unit-years is simply the vertical distance between the demand line of machine center i and that of machine center i + 1. Depending on the location of H in Figures 3.3 and 3.4 the distance can be changed. This distance needs not to be bigger than the distance between one apex of the actual production line of machine center $i + 1$ and the demand line of machine center $i + 1$ no matter where the point H is located. This distance is the maximum distance and is equal to $(\frac{D}{N_{i+1}})(1 - \frac{D}{P_{i+1}})$. For some locations of H the distance between the two demand lines can be much smaller than the maximum distance and the distance cannot be smaller than this for any circumstance. This distance is called minimum distance and its actual quantity will be calculated later.

The vertical distance between the two demand lines, which is the installation inventory of machine center i per year, accordingly, changes within the range of the maximum and minimum distances depending on the location of H in Figures 3.3 and 3.4.

While the determination of the actual location of H is a managerial decision, the following assumption is made in the dissertation: the vertical distance between the two demand lines is the arithematic mean of the maximum distance and the minimum distance. Hence, the installation inventory of machine center i per year is $\frac{1}{2}$(maximum distance + minimum distance) where the maximum distance has been already obtained. The rest of this section deals with finding the minimum distance.

Figure 3.8 shows the available production line at machine center $i + 1$ and the actual production line of machine center $i + 1$ where the two demand lines of machine centers i and $i + 1$ coincide. Also one production starting point of the available production line meets with one of the actual production line at point A as well

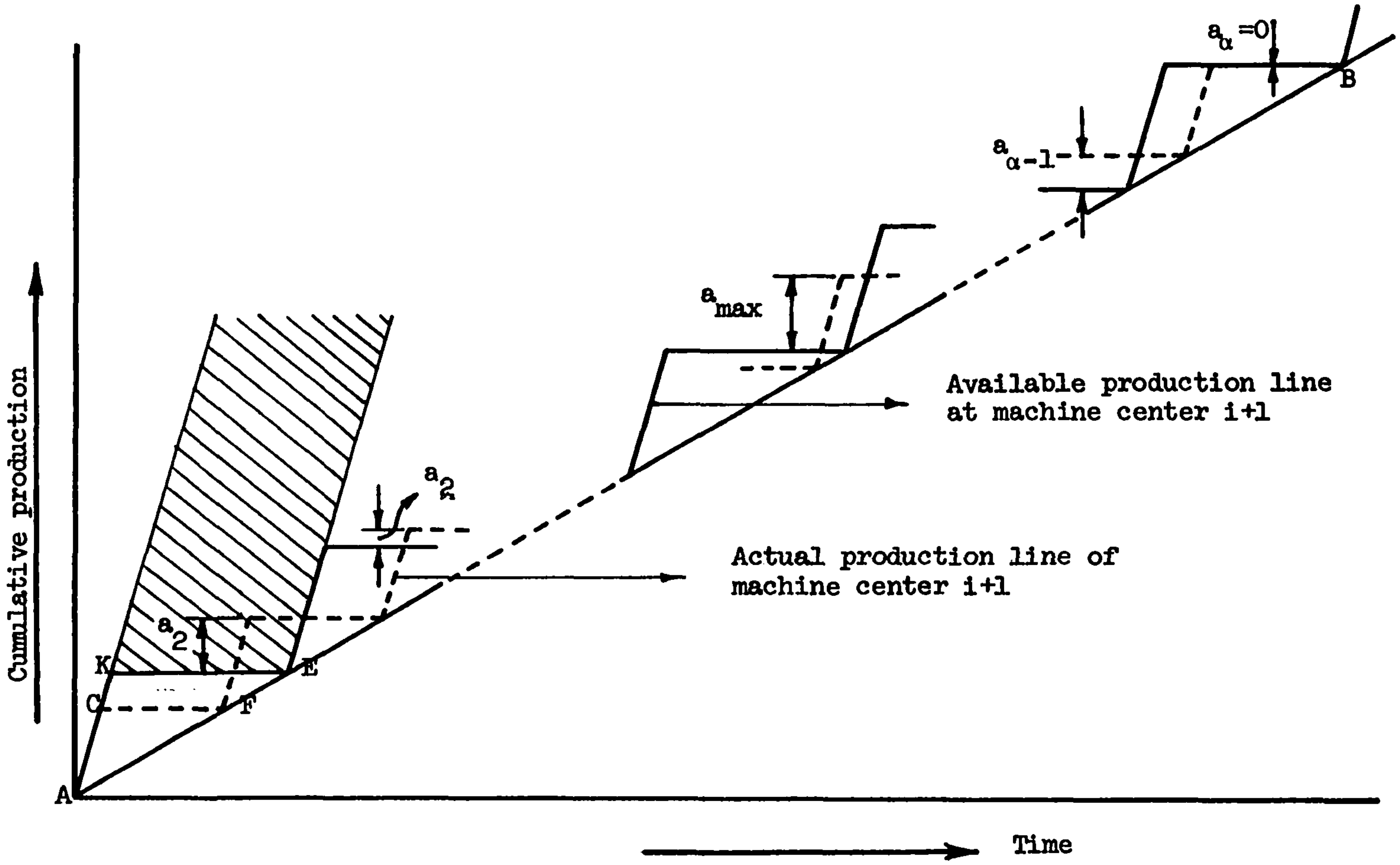

Figure 3.8. Available production line at and actual production line of machine center i+1

as at point B. The portion of the available production line between A and B is composed of α cycles and the portion of the actual production line between A and B is composed of β cycles, where $\frac{N_{i+1}}{N_i} = \frac{\beta}{\alpha}$ and α and β are relatively prime integers.

Consider the line segment KE. It is the first horizontal portion of the available production line at machine center $i + 1$. Unless it is level with CF, which is the first horizontal portion of the actual production line of machine center $i + 1$, there always exists a segment of the actual production line of machine center $i + 1$ in the shaded region in Figure 3.8. The leveling is realized at αth horizontal portion of the available production line and at βth horizontal portion of the actual production line.

a_1 is the vertical distance between KE and the second horizontal portion of the actual production line which passes through the shaded region. Consider a similar shaded region just above the second horizontal portion of the available production line. There will be another segment of the actual production line in that region. a_2 is the vertical distance obtained in similar fashion. Since the αth horizontal portion of the available production line is level with one of the horizontal portions of the actual production line for the first time, the quantities $a_1, a_2, \ldots, a_\alpha$ can be obtained. Note that a_α is always zero.

Suppose that a_{max} is the maximum of $a_1, a_2, \ldots, a_\alpha$. If the whole available production line is moved parallel by an amount a_{max}, the available production line will just cover the actual production line. In no circumstance will a complete coverage be realized by a vertical movement less than a_{max}. Suppose such movement has been realized. Then the vertical distance between the two demand lines will be $a_{max} \cdot (1 - \frac{D}{P_{i+1}})$. Consequently the minimum distance to be found is $a_{max} \cdot (1 - \frac{D}{P_{i+1}})$.

<u>Proposition 1</u>: a_{max} in Figure 3.8 equals $(\frac{D}{N_{i+1}})(1 - \frac{1}{\alpha})$ where $\frac{N_{i+1}}{N_i} = \frac{\alpha}{\beta}$ and α and β are relatively prime integers.

<u>Proof</u>: Let $H_i = \frac{D}{N_i}$ and $H_{i+1} = \frac{D}{N_{i+1}}$. H_i and H_{i+1} are the production quantities per cycle at machine center i and $i + 1$ respectively. It is always possible to express H_i and H_{i+1} in terms of α and β such that $H_i = t \cdot \beta$ and $H_{i+1} = t \cdot \alpha$ for some real number t. Also

$$0 \leq a_1 = \ell_1 \cdot H_{i+1} - H_i < H_{i+1},$$

$$0 \leq a_2 = \ell_2 \cdot H_{i+1} - 2 \cdot H_i < H_{i+1}, \ldots,$$

$$0 \leq a_{\alpha-1} = \ell_{\alpha-1} \cdot H_{i+1} - (\alpha - 1) \cdot H_i < H_{i+1},$$

$$a_\alpha = \ell_\alpha \cdot H_{i+1} - \alpha \cdot H_i = 0 \quad \text{where} \quad \ell_1, \ell_2, \ldots, \ell_\alpha$$

$$\varepsilon \ \{1, 2, \ldots, \beta\} \quad \text{and} \quad \ell_1 \leq \ell_2 \leq \cdots \leq \ell_{\alpha-1} \leq \ell_\alpha = \beta$$

By substituting $t \cdot \beta$ for H_i and $t \cdot \alpha$ for H_{i+1},

$$0 \leq \frac{a_1}{t} = \ell_1 \cdot \alpha - \beta < \alpha,$$

$$0 \leq \frac{a_2}{t} = \ell_2 \cdot \alpha - 2 \cdot \beta < \alpha, \ldots,$$

$$0 \leq \frac{a_{\alpha-1}}{t} = \ell_{\alpha-1} \cdot \alpha - (\alpha - 1) \cdot \beta < \alpha,$$

$$\frac{a_\alpha}{t} = \beta \cdot \alpha - \alpha \cdot \beta = 0.$$

Since α and β are relatively prime integers, each of $\frac{a_1}{t}, \frac{a_2}{t}, \ldots, \frac{a_\alpha}{t}$ will take one unique value among 0, 1, 2, ..., α-1. Consequently $\frac{a_{max}}{t} = \alpha - 1$ and $a_{max} = t(\alpha - 1) = H_{i+1} - \frac{H_{i+1}}{\alpha} = H_{i+1}(1 - \frac{1}{\alpha})$ $= \frac{D}{N_{i+1}} (1 - \frac{1}{\alpha})$. QED.

Since $a_{max} = \frac{D}{N_{i+1}} (1 - \frac{1}{\alpha})$, the minimum distance is

$$a_{max}(1 - \frac{D}{P_{i+1}}) = \frac{D}{N_{i+1}}(1 - \frac{1}{\alpha})(1 - \frac{D}{P_{i+1}}).$$

Hence, the installation inventory of machine center i per year is $\frac{1}{2}$(maximum distance + minimum distance)

$$= \frac{1}{2} \{(\frac{D}{N_{i+1}})(1 - \frac{D}{P_{i+1}}) + (\frac{D}{N_{i+1}})(1 - \frac{1}{\alpha})(1 - \frac{D}{P_{i+1}})\}$$

$$= \frac{1}{2} (\frac{D}{N_{i+1}})(1 - \frac{D}{P_{i+1}})(2 - \frac{1}{\alpha}).$$

4. The WIP inventory of 1 part and M machine centers and its holding cost

From Section 2 the production inventory of machine center i per year is $\frac{1}{2} \cdot \frac{D}{N_i} \cdot (1 - \frac{D}{P_i})$. Also from Section 3, the installation inventory of machine center i per year is $\frac{1}{2} (\frac{D}{N_{i+1}})(1 - \frac{D}{P_{i+1}})(2 - \frac{1}{\alpha})$.

Now, it is possible to calculate the WIP inventory between machine centers i and $i + 1$ per year from equation 1 in Section 1.

(WIP inventory between machine centers i and $i + 1$ per year)

$$= \frac{1}{2} \cdot \frac{D}{N_i} \cdot \left(1 - \frac{D}{P_i}\right) + \frac{1}{2}\left(\frac{D}{N_{i+1}}\right)\left(1 - \frac{D}{P_{i+1}}\right)\left(2 - \frac{1}{\alpha}\right) -$$

$$- \frac{1}{2} \cdot \frac{D}{N_{i+1}} \cdot \left(1 - \frac{D}{P_{i+1}}\right) .$$

Since the dollar value of the part between machine centers i and $i + 1$ is assumed to be V_i from assumption 13, the yearly inventory holding cost for the above WIP inventory is (WIP inventory between machine center i and $i + 1$ per year) . V_i . I.

In calculating the total WIP inventory per year associated with M machine centers, the following symbols are defined.

WIP_i: The WIP inventory between machine centers i and $i + 1$ per year.

PIY_i: The production inventory of machine center i per year.

IIY_i: The installation inventory of machine center i per year.

α_i: $\frac{N_{i+1}}{N_i} = \frac{\beta_{i+1}}{\alpha_i}$ where α_i and β_{i+1} are relatively prime integers

The total WIP inventory per year between machine centers 1 and M is the sum of the WIP inventories between each pair of machine centers, i.e., 1 and 2, 2 and 3, ..., M-1 and M. Since the last machine center, M, produces cyclically and the demand rate at the shipping area is a straight line, there is some WIP inventory between machine center M and the shipping area. The amount of this inventory is PIY_M. The total WIP inventory of this production system is, then the inventory between machine centers 1 and M plus PIY_M.

$$\begin{aligned}(\text{Total WIP inventory per year}) &= (PIY_1 + IIY_1 - PIY_2) \\ &+ (PIY_2 + IIY_2 - PIY_3) + \dots \\ &+ (PIY_{M-1} + IIY_{M-1} - PIY_M) + PIY_M \\ &= PIY_1 + \sum_{i=1}^{M-1} IIY_i \end{aligned} \tag{3.2}$$

Since $V_1, V_2, \dots, V_M$ are known, the yearly inventory holding cost for the above total WIP inventory can be calculated.

$$\begin{aligned}(\text{Total holding cost per year}) &= (PIY_1 + IIY_1 - PIY_2) \cdot V_1 \cdot I \\ &+ (PIY_2 + IIY_2 - PIY_3) \cdot V_2 \cdot I + \dots \\ &+ (PIY_{M-1} + IIY_{M-1} - PIY_M)\, V_{M-1} \cdot I + PIY_M \cdot V_M \cdot I \\ &= PIY_1 \cdot V_1 \cdot I + \sum_{i=2}^{M} PIY_i (V_i - V_{i-1}) \cdot I \\ &+ \sum_{i=1}^{M-1} IIY_i \cdot V_i \cdot I \end{aligned} \tag{3.3}$$

By substituting $\frac{1}{2} \cdot \frac{D}{N_i} \cdot (1 - \frac{D}{P_i})$ for PIY_i and $\frac{1}{2} \cdot (\frac{D}{N_{i+1}})(1 - \frac{D}{P_{i+1}})(2 - \frac{1}{\alpha_i})$ for IIY_i in equations 2 and 3,

$$\begin{aligned}(\text{Total WIP inventory per year}) &= \frac{1}{2} \cdot \frac{D}{N_1} \cdot (1 - \frac{D}{P_1}) \\ &+ \sum_{i=1}^{M-1} \frac{1}{2} \cdot (\frac{D}{N_{i+1}})(1 - \frac{D}{P_{i+1}})(2 - \frac{1}{\alpha_i}) \end{aligned} \tag{3.4}$$

$$(\text{Total holding cost per year}) = \sum_{i=1}^{M} \frac{1}{2} \cdot \frac{D}{N_i} (1 - \frac{D}{P_i}) V_i \cdot I$$

$$+ \sum_{i=1}^{M-1} \frac{1}{2} \cdot (\frac{D}{N_{i+1}})(1 - \frac{D}{P_{i+1}})(1 - \frac{1}{\alpha_i}) \cdot V_i \cdot I \qquad (3.5)$$

Usually it is true that $V_1 < V_2 < V_3 \cdots < V_M$ because the unit value of the part will increase by some positive amount every time it passes through a machine center. However, equation 3.5 is still valid for the case where there is no such restriction as above on the relationships among $V_1, V_2, \ldots, V_M$.

5. The relaxation of the assumption that $P_i = P_{i+1}$

In Section 1 it is assumed that the production rate of machine center i is the same as that of machine center $i + 1$. Because of this assumption, $P_1 = P_2 = \ldots = P_M$ in equations 3.4 and 3.5. If this assumption is relaxed, a different result occurs. If the production rates of all M machine centers are such that $P_1 \geq P_2 \geq, \ldots, \geq P_M$, equations 3.4 and 3.5 are still valid. To show the reason for this, a similar picture to Figure 3.8 is drawn in Figure 3.9. The difference between Figure 3.8 and Figure 3.9 is that $P_i > P_{i+1}$ and the available production line at machine center $i + 1$ has been moved along the line AK in Figure 3.8 by an amount equal its total vertical movement, a_{max}.

It is obvious from Figure 3.9 that the available production line at machine center $i + 1$ can cover the actual production line by the minimum distance, $a_{max} \cdot (1 - \frac{D}{P_{i+1}})$, when $P_i > P_{i+1}$.

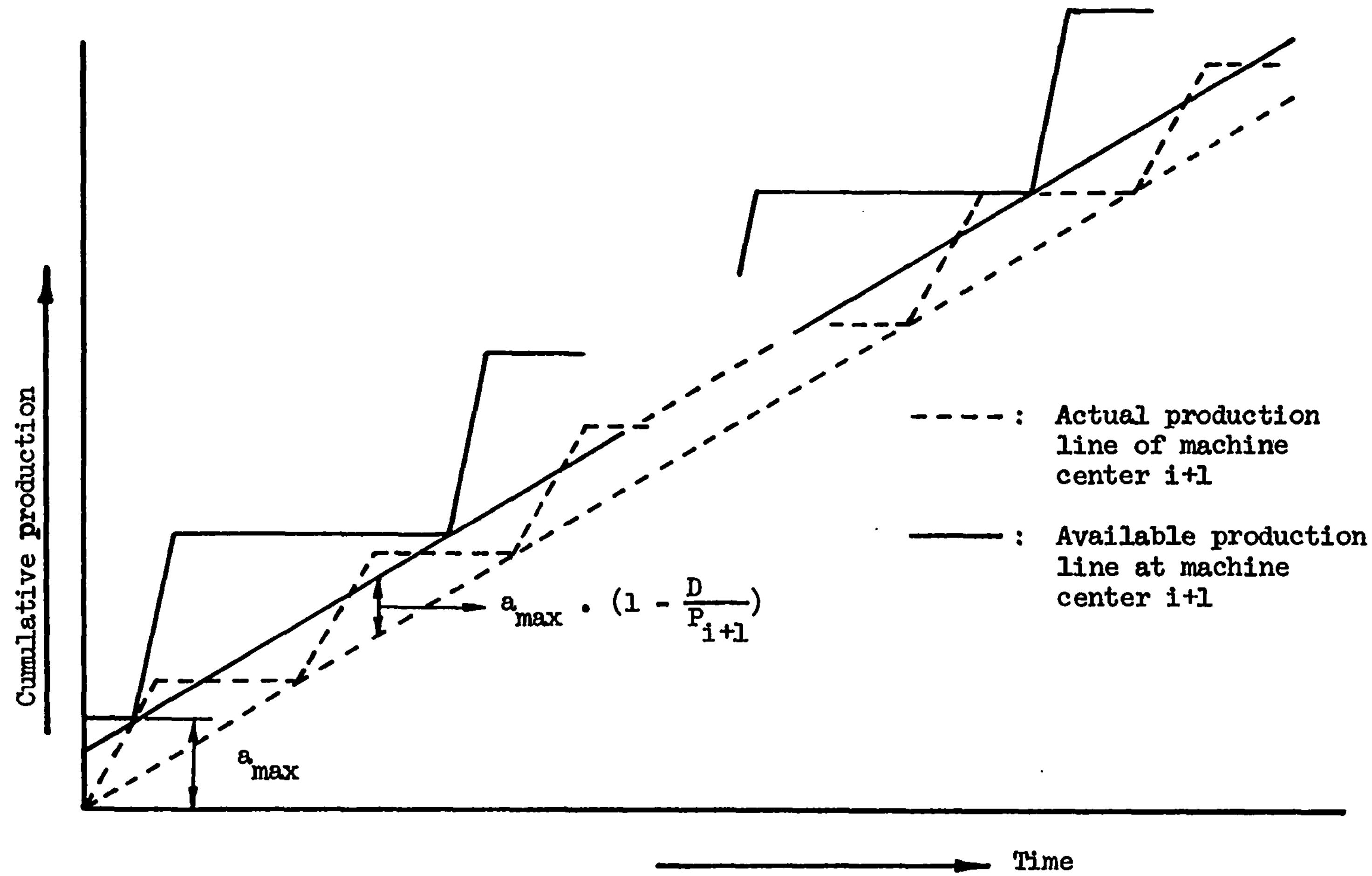

Figure 3.9. Available production line at and actual production line of machine center $i+1$ when $P_i > P_{i+1}$

If $P_i < P_{i+1}$, the minimum distance, $a_{max} \cdot (1 - \frac{D}{P_{i+1}})$, is not sufficient for complete coverage and this is shown in Figure 3.10.

To realize a complete coverage the available production line at machine center $i + 1$ is moved horizontally to the left as much as CG in Figure 3.10. Once this movement has been realized, the minimum distance between the demand line of machine center i and that of $i + 1$ will be bigger than $a_{max} \cdot (1 - \frac{D}{P_{i+1}})$ by $CG \cdot D$. From Figure 3.10, $CG = (\frac{D}{N_{i+1}})(\frac{1}{\alpha_i})(\frac{1}{P_i} - \frac{1}{P_{i+1}})$. Hence the minimum distance will be $a_{max} \cdot (1 - \frac{D}{P_{i+1}}) + (\frac{D}{N_{i+1}})(\frac{1}{\alpha_i})(\frac{1}{P_i} - \frac{1}{P_{i+1}}) \cdot D$.

In Figure 3.10, the installation inventory of machine center i will be $\frac{1}{2}$ (maximum distance + minimum distance) $= \frac{1}{2}\{(\frac{D}{N_{i+1}})(1 - \frac{D}{P_{i+1}})$

$$+ (\frac{D}{N_{i+1}})(1 - \frac{1}{\alpha_i})(1 - \frac{D}{P_{i+1}})$$

$$+ (\frac{D}{N_{i+1}})(\frac{1}{\alpha_i})(\frac{1}{P_i} - \frac{1}{P_{i+1}}) \cdot D\}$$

$$= \frac{1}{2}(\frac{D}{N_{i+1}})(1 - \frac{D}{P_{i+1}})(2 - \frac{1}{\alpha_i})$$

$$+ \frac{1}{2}(\frac{D}{N_{i+1}})(\frac{D}{\alpha_i})(\frac{1}{P_i} - \frac{1}{P_{i+1}}).$$

To summarize, the installation inventory of machine center i can be calculated as follows:

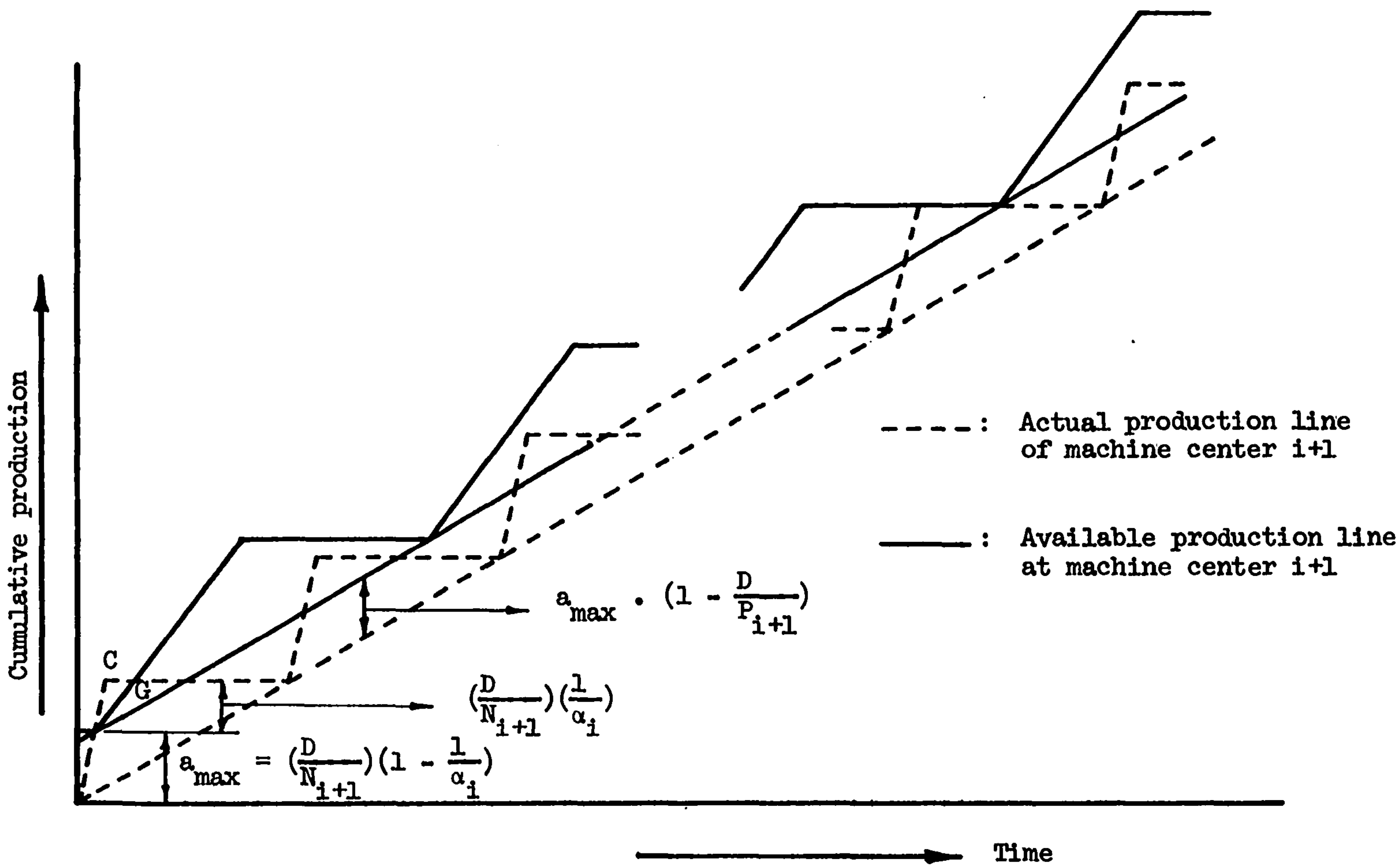

Figure 3.10. Available production line at and actual production line of machine center i+1 when $P_i < P_{i+1}$

P_i and P_{i+1}	Maximum Distance	Minimum Distance	The Installation Inventory of Machine Center i Per Year
$P_i \geq P_{i+1}$	$(\frac{D}{N_{i+1}})(1-\frac{D}{P_{i+1}})$	$(\frac{D}{N_{i+1}})(1-\frac{D}{P_{i+1}})(1-\frac{1}{\alpha_i})$	$\frac{1}{2}$(Max + Min)
$P_i < P_{i+1}$	$(\frac{D}{N_{i+1}})(1-\frac{D}{P_{i+1}})$	$(\frac{D}{N_{i+1}})(1-\frac{D}{P_{i+1}})(1-\frac{1}{\alpha_i}) + (\frac{D}{N_{i+1}})(\frac{D}{\alpha_i})(\frac{1}{P_i} - \frac{1}{P_{i+1}})$	$\frac{1}{2}$(Max + Min)

To accommodate the above two cases in one equation form a function $\delta(.)$ is defined as follows:

$\delta(x) = 0$ when $x \leq 0$

$\delta(x) = 1$ when $x > 0$.

By utilizing $\delta(.)$, the installation inventory of machine center i is

$$\frac{1}{2}(\frac{D}{N_{i+1}})(1 - \frac{D}{P_{i+1}})(2 - \frac{1}{\alpha_i})$$

$$+ \frac{1}{2}(\frac{D}{N_{i+1}})(\frac{D}{\alpha_i})(\frac{1}{P_i} - \frac{1}{P_{i+1}}) \cdot \delta(P_{i+1} - P_i).$$

Equations 3.4 and 3.5 can be modified by substituting

$\frac{1}{2} \cdot \frac{D}{N_i} \cdot (1 - \frac{D}{P_i})$ for PIY_i and $\frac{1}{2}(\frac{D}{N_{i+1}})(1 - \frac{D}{P_{i+1}})(2 - \frac{1}{\alpha_i})$

$$+ \frac{1}{2} \left(\frac{D}{N_{i+1}}\right)\left(\frac{D}{\alpha_i}\right)\left(\frac{1}{P_i} - \frac{1}{P_{i+1}}\right) \cdot \delta(P_{i+1} - P_i)$$

for IIY_i in equations 3.2 and 3.3 as follows.

$$(\text{Total WIP inventory per year}) = \frac{1}{2} \cdot \frac{D}{N_1} \cdot \left(1 - \frac{D}{P_1}\right)$$

$$+ \sum_{i=1}^{M-1} \left\{\frac{1}{2} \cdot \left(\frac{D}{N_{i+1}}\right)\left(1 - \frac{D}{P_{i+1}}\right)\left(2 - \frac{1}{\alpha_i}\right)\right.$$

$$\left. + \frac{1}{2} \cdot \left(\frac{D}{N_{i+1}}\right)\left(\frac{D}{\alpha_i}\right)\left(\frac{1}{P_i} - \frac{1}{P_{i+1}}\right) \cdot \delta(P_{i+1} - P_i)\right\} \qquad (3.6)$$

$$(\text{Total holding cost per year}) = \sum_{i=1}^{M} \frac{1}{2} \cdot \frac{D}{N_i} \cdot \left(1 - \frac{D}{P_i}\right) \cdot V_i \cdot I$$

$$+ \sum_{i=1}^{M-1} \left\{\frac{1}{2} \cdot \left(\frac{D}{N_{i+1}}\right)\left(1 - \frac{D}{P_{i+1}}\right)\left(1 - \frac{1}{\alpha_i}\right)\right.$$

$$\left. + \frac{1}{2} \cdot \left(\frac{D}{N_{i+1}}\right)\left(\frac{D}{\alpha_i}\right)\left(\frac{1}{P_i} - \frac{1}{P_{i+1}}\right) \cdot \delta(P_{i+1} - P_i)\right\} \cdot V_i \cdot I$$

$$(3.7)$$

6. The relaxation of assumption 7

In previous sections it is assumed that the part being produced is infinitely divisible, such as liquid or powder, and once it has passed through a machine center it is ready to be sent to a subsequent machine center. These assumptions are justified when demand is very large, unit processing time is very small and each individual unit can be sent to a next operation as soon as its present operation is finished. However, it is possible that a significant amount of time is required to finish each operation for each unit. Also it is a normal practice to move parts in containers such as tote boxes or skids. In this situation each individual unit is not ready to be moved to a next machine center until a certain number of units have filled each container.

The objective of this section is to modify equations 3.6 and 3.7 in such a way that the effects of using containers on the WIP inventory and its holding cost can be taken into consideration.

Consider Figure 3.11 in which the total units produced for one cycle at machine center i are moved with 3 containers. Attention should be given to the difference between the amount produced during one cycle and the amount filled in one container. The former is called one lot and the latter is called one box. One lot is sometimes called Economic Order Quantity, Economic Production Quantity or Economic Batch Size in the literature.

In Figure 3.11, the triangular ACE represents one box. No single box unit can be used or moved to another machine center until

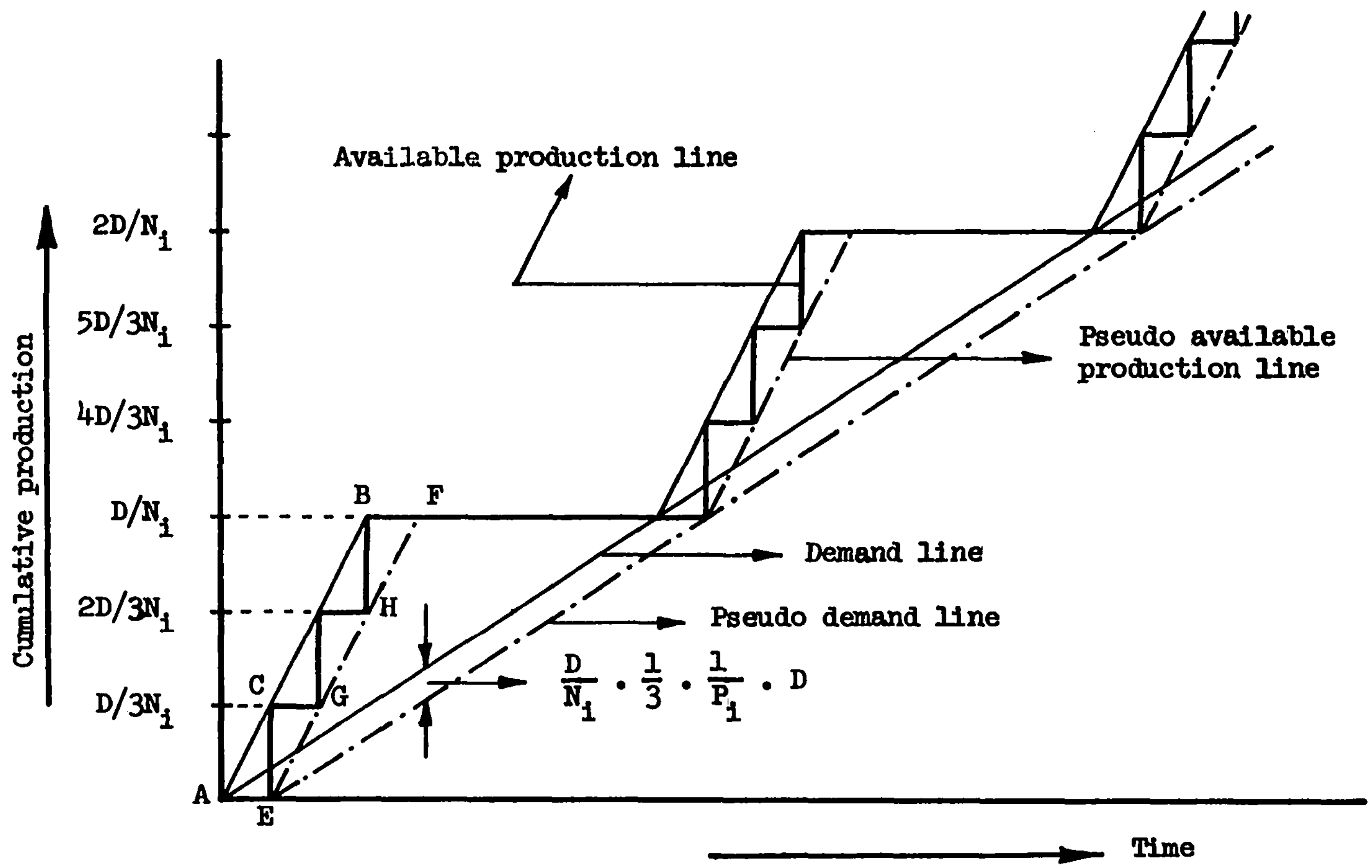

Figure 3.11. Available and pseudo available production lines at machine center i+1

time AE has elapsed and an entire box quantity, $\frac{D}{3N_i}$, has been processed. Because of this restriction, the available production line in Figure 3.11 represents the availability of incoming material flow to machine center $i + 1$. Accordingly the available production line is the one which should cover completely the actual production line of machine center $i + 1$.

When $P_i \geq P_{i+1}$, the complete coverage of the actual production line of machine center $i + 1$ by the available production line at machine center $i + 1$ is equivalent to the complete coverage by the pseudo available production line. This is due to the vertexes E, G, and H in Figure 3.11. The logical way to handle this situation is to let the pseudo available production line take the role of the available production line in the development of the WIP inventory calculation in previous sections. Since the vertical distance between the demand line and the pseudo demand line is $\frac{D}{N_i} \cdot \frac{1}{3} \cdot \frac{1}{P_i} \cdot D$, the only necessary modification for this situation is to shift both the maximum distance and the minimum distance upwards by $\frac{D}{N_i} \cdot \frac{1}{3} \cdot \frac{1}{P_i} \cdot D$. Then the installation inventory of machine center i per year will be $\frac{1}{2}\left(\frac{D}{N_{i+1}}\right)\left(1 - \frac{D}{P_{i+1}}\right)\left(2 - \frac{1}{\alpha_i}\right) + \frac{D}{N_i} \cdot \frac{1}{3} \cdot \frac{1}{P} \cdot D$.

The production inventory of machine i per year is the same as before.

A complication arises when $P_i < P_{i+1}$. In this situation the pseudo available production line of machine center i can no longer take the role of the available production line as before. This is shown in Figure 3.12.

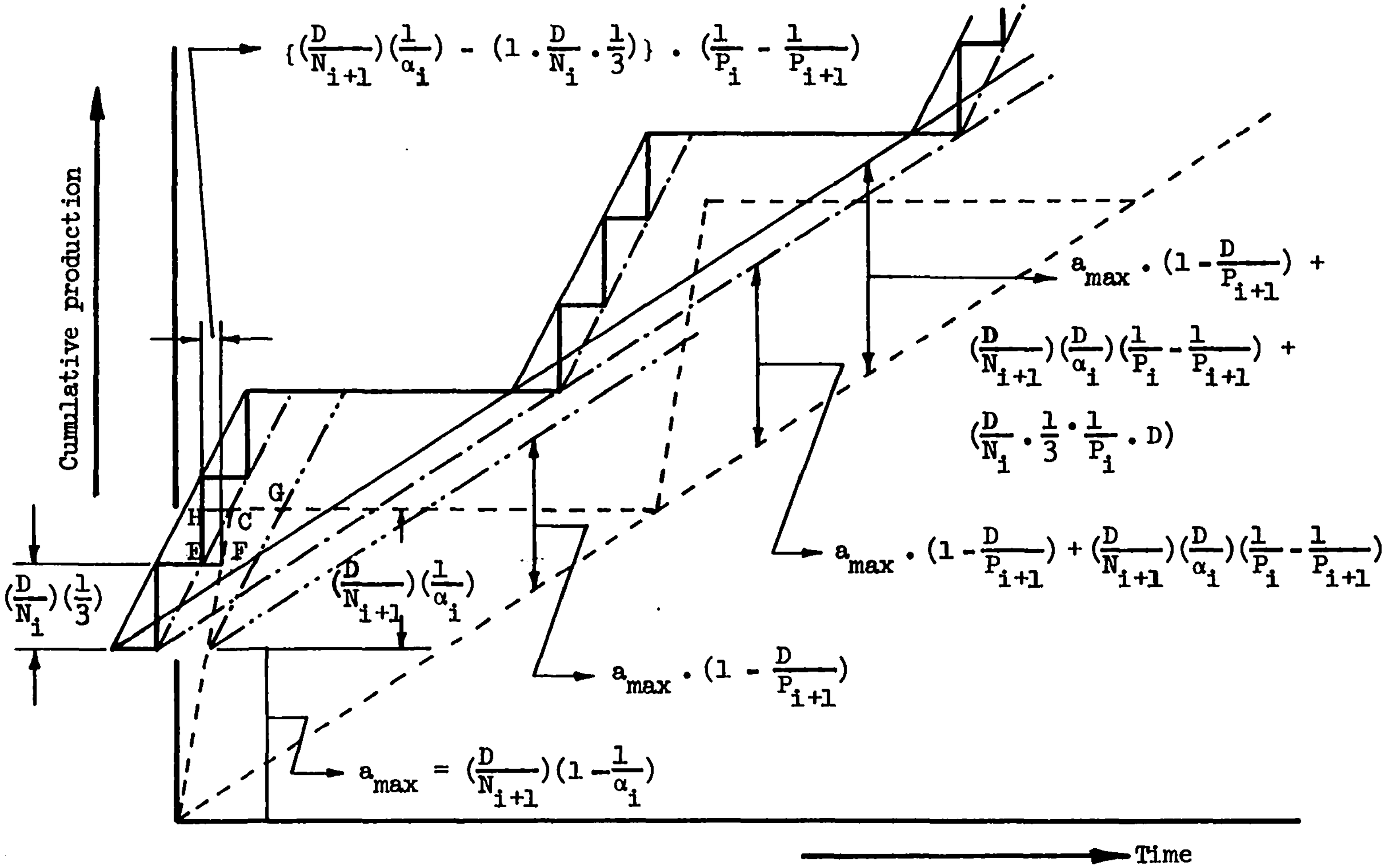

Figure 3.12. Pseudo available production line at machine center i+1 and actual production line of machine center i+1 when $P_i < P_{i+1}$

In Figure 3.12 the double dotted line (-..-) represents the original position of the pseudo available production line at machine center $i + 1$ after its upward movement by the vertical distance a_{max}. This line is similar to the solid line in Figure 3.10. The single dotted line (-.-) represents the position after the double dotted line has been moved horizontally to the left by the distance CG. However, the horizontal movement CG is more than necessary by EF and this is the reason why the pseudo available production line cannot play the role of the available production line as before.

The maximum distance for the calculation of installation inventory of machine center i for this situation is $(\frac{D}{N_{i+1}})(1 - \frac{D}{P_{i+1}}) + (\frac{D}{N_i} \cdot \frac{1}{3} \cdot \frac{1}{P_i} \cdot D)$. But the minimum distance is less than

$$a_{max} \cdot (1 - \frac{D}{P_{i+1}}) + (\frac{D}{N_{i+1}}) \ (\frac{D}{\alpha_i}) \ (\frac{1}{P_i} - \frac{1}{P_{i+1}}) + (\frac{D}{N_i} \cdot \frac{1}{3} \cdot \frac{1}{P_i} \cdot D)$$

in Figure 3.12 due to the potential right movement of the pseudo production line of machine center i by EF. The potential decrement is EF . D. Since $EH = (\frac{D}{N_{i+1}})(\frac{1}{\alpha_i}) - (1 \cdot \frac{D}{N_i} \cdot \frac{1}{3})$ and

$EF = EH \cdot (\frac{1}{P_i} - \frac{1}{P_{i+1}})$, $EF = \{(\frac{D}{N_{i+1}})(\frac{1}{\alpha_i}) - (1 \cdot \frac{D}{N_i} \cdot \frac{1}{3})\} \cdot (\frac{1}{P_i} - \frac{1}{P_{i+1}})$.

Consequently the correct minimum distance will be

$$a_{max} \cdot (1 - \frac{D}{P_{i+1}}) + (\frac{D}{N_{i+1}})(\frac{D}{\alpha_i})(\frac{1}{P_i} - \frac{1}{P_{i+1}}) + (\frac{D}{N_i} \cdot \frac{1}{3} \cdot \frac{1}{P_i} \cdot D)$$

$$- \{(\frac{D}{N_{i+1}})(\frac{1}{\alpha_i}) - (1 \cdot \frac{D}{N_i} \cdot \frac{1}{3})\} \cdot (\frac{1}{P_i} - \frac{1}{P_{i+1}}) \cdot D.$$

The number,"1", in the term $(1 \cdot \frac{D}{N_i} \cdot \frac{1}{3})$ is kept to emphasize the multiplier significance. In Figure 3.12, the distance $(\frac{D}{N_{i+1}}) \cdot (\frac{1}{\alpha_i})$ is about 1.6 times as big as the distance $(\frac{D}{N_i} \cdot \frac{1}{3})$. The number, 1, is the integer part of 1.6.

It is possible to develop a general equation form of the minimum distance for a case where the number of containers used at machine center i is n_i. Let K_i be the integer part of

$$\left(\frac{\frac{D}{N_{i+1}} \cdot \frac{1}{\alpha_i}}{\frac{D}{N_i} \cdot \frac{1}{n_i}}\right) = \left(\frac{n_i}{\beta_{i+1}}\right)$$

where $\frac{N_{i+1}}{N_i} = \frac{\beta_{i+1}}{\alpha_i}$ and α_i and β_{i+1} are relatively prime integers.

When $(\frac{n_i}{\beta_{i+1}})$ is integer itself, let K_i be $(\frac{n_i}{\beta_{i+1}}) - 1$. Then the minimum distance will be

$$\begin{aligned} & a_{max} \cdot (1 - \frac{D}{P_{i+1}}) + (\frac{D}{N_{i+1}})(\frac{D}{\alpha_i})(\frac{1}{P_i} - \frac{1}{P_{i+1}}) \\ & + (\frac{D}{N_i} \cdot \frac{1}{n_i} \cdot \frac{1}{P_i} \cdot D) - \{(\frac{D}{N_{i+1}})(\frac{1}{\alpha_i}) \\ & - (K_i \cdot \frac{D}{N_i} \cdot \frac{1}{n_i})\} \cdot (\frac{1}{P_i} - \frac{1}{P_{i+1}}) \cdot D \\ & = (\frac{D}{N_{i+1}})(1 - \frac{D}{P_{i+1}})(1 - \frac{1}{\alpha_i}) + \frac{D}{N_i} \cdot \frac{1}{n_i} \cdot \frac{1}{P_i} \cdot D \\ & + K_i \cdot \frac{D}{N_i} \cdot \frac{1}{n_i} \cdot (\frac{1}{P_i} - \frac{1}{P_{i+1}}) \cdot D \end{aligned}$$

The development in this section can be summarized as follows.[1]

P_i and P_{i+1}	Maximum Distance	Minimum Distance	The Installation Inventory of Machine Center i Per Year
$P_i \geq P_{i+1}$	$(\frac{D}{N_{i+1}})(1-\frac{D}{P_{i+1}}) + \frac{D}{N_i} \cdot \frac{1}{n_i} \cdot \frac{1}{P_i} \cdot D$	$(\frac{D}{N_{i+1}})(1-\frac{D}{P_{i+1}})(1-\frac{1}{\alpha_i}) + \frac{D}{N_i} \cdot \frac{1}{n_i} \cdot \frac{1}{P_i} \cdot D$	$\frac{1}{2}$ (Max + Min)
$P_i < P_{i+1}$	$(\frac{D}{N_{i+1}})(1-\frac{D}{P_{i+1}}) + \frac{D}{N_i} \cdot \frac{1}{n_i} \cdot \frac{1}{P_i} \cdot D$	$(\frac{D}{N_{i+1}})(1-\frac{D}{P_{i+1}})(1-\frac{1}{\alpha_i}) + \frac{D}{N_i} \cdot \frac{1}{n_i} \cdot \frac{1}{P_i} \cdot D + K_i \cdot \frac{D}{N_i} \cdot \frac{1}{n_i} \cdot (\frac{1}{P_i} - \frac{1}{P_{i+1}}) \cdot D$	$\frac{1}{2}$ (Max + Min)

By using the $\delta(.)$ in Section 5, the installation inventory of machine center i is

$$\frac{1}{2}\left(\frac{D}{N_{i+1}}\right)\left(1 - \frac{D}{P_{i+1}}\right)\left(2 - \frac{1}{\alpha_i}\right) + \frac{D}{N_i} \cdot \frac{D}{n_i} \cdot \frac{1}{P_i}$$

$$+ \frac{1}{2} K_i \frac{D}{N_i} \cdot \frac{D}{n_i} \cdot \left(\frac{1}{P_i} - \frac{1}{P_{i+1}}\right) \cdot \delta(P_{i+1} - P_i).$$

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[1]The development is only valid when $n_i \geq \frac{\beta_{i+1} \cdot D(P_{i+1} - P_i)}{2(P_i - D)(P_{i+1})}$.

Since P_i and P_{i+1} are much larger than D and $\beta_{i+1} \leq \bar{N}_{i+1}$, this condition will be satisfied in most cases. Refer to Appendix A for the condition.

It is worthwhile to observe the installation inventory of machine center i as n_i changes.

(Case A) $n_i \to \infty$. Then $K_i = \frac{n_i}{\beta_{i+1}}$.

$$\text{Installation Inventory} = \frac{1}{2}\left(\frac{D}{N_{i+1}}\right)\left(1 - \frac{D}{P_{i+1}}\right)\left(2 - \frac{1}{\alpha_i}\right)$$

$$+ \frac{D}{N_i} \cdot \frac{D}{n_i} \cdot \frac{1}{P_i}$$

$$+ \frac{1}{2} \cdot \frac{n_i}{\beta_{i+1}} \cdot \frac{D}{N_i} \cdot \frac{D}{n_i} \cdot \left(\frac{1}{P_i} - \frac{1}{P_{i+1}}\right) \cdot \delta(P_{i+1} - P_i)$$

$$= \frac{1}{2}\left(\frac{D}{N_{i+1}}\right)\left(1 - \frac{D}{P_{i+1}}\right)\left(2 - \frac{1}{\alpha_i}\right)$$

$$+ \frac{1}{2}\left(\frac{D}{N_{i+1}}\right)\left(\frac{D}{\alpha_i}\right)\left(\frac{1}{P_i} - \frac{1}{P_{i+1}}\right) \cdot \delta(P_{i+1} - P_i).$$

Here

$\beta_{i+1} \cdot N_i = \alpha_i \cdot N_{i+1}$. This result is consistent with that of Section 5. The reason is the fact that as $n_i \to \infty$, the pseudo available production line will eventually coincide with the available production line and the situation will be reduced to the one in Section 5. Accordingly, assumption 7 is equivalent to setting $n_i \to \infty$.

(Case B) $n_i = 1$. Then $K_i = 0$ since $\beta_{i+1} \geq 1$.

$$\text{Installation Inventory} = \frac{1}{2}\left(\frac{D}{N_{i+1}}\right)\left(1 - \frac{D}{P_{i+1}}\right)\left(2 - \frac{1}{\alpha_i}\right) + \frac{D}{N_i} \cdot \frac{D}{P_i}.$$

(Case C) $n_i \leq \beta_{i+1}$. Then $K_i = 0$.

$$\text{Installation Inventory} = \frac{1}{2}\left(\frac{D}{N_{i+1}}\right)\left(1 - \frac{D}{P_{i+1}}\right)\left(2 - \frac{1}{\alpha_i}\right) + \frac{D}{N_i} \cdot \frac{D}{P_i} \cdot \frac{1}{n_i}$$

(Case D) $n_i = \frac{D}{N_i}$. In this case the number of containers is the same as the number of total units of one lot. This means that individual units can be sent to a subsequent machine center as soon as its present operation is completed

i) when $P_i \geq P_{i+1}$, then

$$\text{Installation Inventory} = \frac{1}{2}\left(\frac{D}{N_{i+1}}\right)\left(1 - \frac{D}{P_{i+1}}\right)\left(2 - \frac{1}{\alpha_i}\right) + \frac{D}{P_i}$$

ii) when $P_i < P_{i+1}$, it is difficult to calculate the installation inventory accurately. Under the assumption of large D,

$$K_i = \frac{n_i}{\beta_{i+1}} = \frac{D}{N_i \cdot \beta_{i+1}}.$$

$$\text{Installation Inventory} = \frac{1}{2}\left(\frac{D}{N_{i+1}}\right)\left(1 - \frac{D}{P_{i+1}}\right)\left(2 - \frac{1}{\alpha_i}\right) + \frac{D}{P_i}$$

$$+ \frac{1}{2}\left(\frac{D}{N_i \cdot \beta_{i+1}}\right)\left(\frac{D}{N_i}\right)\left(\frac{D}{n_i}\right) \cdot \left(\frac{1}{P_i} - \frac{1}{P_{i+1}}\right) \cdot \delta(P_{i+1} - P_i)$$

(Case E) When $\frac{n_i}{\beta_{i+1}} = k_i$ where k_i is a positive integer, then

$K_i = k_i - 1.$

$$\text{Installation Inventory} = \frac{1}{2}\left(\frac{D}{N_{i+1}}\right)\left(1 - \frac{D}{P_{i+1}}\right)\left(2 - \frac{1}{\alpha_i}\right)$$

$$+ \frac{D}{N_i} \cdot \frac{D}{k_i \cdot \beta_{i+1}} \cdot \frac{1}{P_i}$$

$$+ \frac{1}{2}(k_i - 1) \cdot \frac{D}{N_i} \cdot \frac{D}{k_i \cdot \beta_{i+1}} \cdot \left(\frac{1}{P_i} - \frac{1}{P_{i+1}}\right) \cdot \delta(P_{i+1} - P_i)$$

$$= \frac{1}{2}\left(\frac{D}{N_{i+1}}\right)\left(1 - \frac{D}{P_{i+1}}\right)\left(2 - \frac{1}{\alpha_i}\right)$$

$$+ \frac{1}{k_i} \cdot \frac{D}{N_{i+1}} \cdot \frac{D}{\alpha_i} \cdot \frac{1}{P_i}$$

$$+ \frac{1}{2} \cdot \frac{(k_i - 1)}{k_i} \cdot \left(\frac{D}{N_{i+1}}\right)\left(\frac{D}{\alpha_i}\right)\left(\frac{1}{P_i} - \frac{1}{P_{i+1}}\right) \cdot \delta(P_{i+1} - P_i).$$

Among the above 5 cases considered, Case E seems to be the most realistic one. All subsequent reference to the installation inventory of machine center i will take the form of Case E.

Equations 3.6 and 3.7 can be modified by substituting $\frac{1}{2} \cdot \frac{D}{N_i} \cdot \left(1 - \frac{D}{P_i}\right)$ for PIY_i and $\frac{1}{2}\left(\frac{D}{N_{i+1}}\right)\left(1 - \frac{D}{P_{i+1}}\right)\left(2 - \frac{1}{\alpha_i}\right)$

$$+ \frac{1}{k_i} \cdot \frac{D}{N_{i+1}} \cdot \frac{D}{\alpha_i} \cdot \frac{1}{P_i} + \frac{1}{2} \cdot \left(\frac{k_i - 1}{k_i}\right)\left(\frac{D}{N_{i+1}}\right)\left(\frac{D}{\alpha_i}\right)\left(\frac{1}{P_i} - \frac{1}{P_{i+1}}\right) \cdot \delta(P_{i+1} - P_i)$$

for IIY_i in equations 3.2 and 3.3 with one minor adjustment. The WIP inventory between machine center M and the shipping area is PIY_M in equations 3.2 and 3.3. When the number of containers used at machine center M is n_M, the quantity of this inventory will be $PIY_M + \frac{D}{N_M} \cdot \frac{D}{n_M} \cdot \frac{1}{P_M}$. Equations 3.2 and 3.3 are only valid for the case when $n_M = \infty$. By taking into consideration this adjustment, equations 3.6 and 3.7 are modified as follows:

$$\text{(Total WIP inventory per year)} = PIY_1 + \sum_{i=1}^{M-1} IIY_i + \frac{D}{N_M} \cdot \frac{D}{n_M} \cdot \frac{1}{P_M}$$

$$= \frac{1}{2}\frac{D}{N_1}\left(1 - \frac{D}{P_1}\right) + \sum_{i=1}^{M-1} \left\{\frac{1}{2}\left(\frac{D}{N_{i+1}}\right)\left(1 - \frac{D}{P_{i+1}}\right)\left(2 - \frac{1}{\alpha_i}\right)\right.$$

$$+ \frac{1}{k_i} \cdot \frac{D}{N_{i+1}} \cdot \frac{D}{\alpha_i} \cdot \frac{1}{P_i}$$

$$\left. + \frac{1}{2}\left(\frac{k_i - 1}{k_i}\right)\frac{D}{N_{i+1}} \cdot \frac{D}{\alpha_i}\left(\frac{1}{P_i} - \frac{1}{P_{i+1}}\right) \cdot \delta(P_{i+1} - P_i)\right\} \tag{3.8}$$

$$\text{(Total holding cost per year)} = PIY_1 \cdot V_1 \cdot I + \sum_{i=2}^{M} PIY_i(V_i - V_{i-1}) \cdot I$$

$$+ \sum_{i=1}^{M-1} IIY_i \cdot V_i \cdot I + \frac{D}{N_M} \cdot \frac{D}{n_M} \cdot \frac{1}{P_M} \cdot V_M \cdot I$$

$$= \sum_{i=1}^{M} \frac{1}{2}\frac{D}{N_i}\left(1 - \frac{D}{P_i}\right) \cdot V_i \cdot I$$

$$+ \sum_{i=1}^{M-1} \frac{1}{2} \cdot \frac{D}{N_{i+1}} \left\{\left(1 - \frac{D}{P_{i+1}}\right)\left(1 - \frac{1}{\alpha_i}\right) + \frac{2}{k_i}\frac{D}{\alpha_i} \cdot \frac{1}{P_i}\right.$$

$$\left. + \left(\frac{k_i - 1}{k_i}\right)\left(\frac{D}{\alpha_i}\right)\left(\frac{1}{P_i} - \frac{1}{P_{i+1}}\right) \delta(P_{i+1} - P_i)\right\} V_i \cdot I$$

$$+ \frac{D}{N_M} \cdot \frac{D}{N_M} \cdot \frac{1}{P_M} \cdot V_M \cdot I \qquad (3.9)$$

B. Optimum Cycles of Machine Centers without Capacity Constraints

1. Introduction

Section B deals with finding the optimum number of cycles per year for each machine center under the assumption that the available machine capacity is infinite. The quantity to be minimized is the total cost which is the setup cost plus inventory holding cost. In expressing the inventory holding cost, equation 3.9 is used. The first case to be dealt with is the one where there are M machine centers and 1 part. No backtracking is allowed in the operation sequence of the part, and an optimum solution is found by using the Dynamic Programming. The second case to be dealt with is a general case where there are M machine centers and N parts. Backtracking is allowed in the operation sequence of the part. Once the number of different parts is bigger than 1 or backtracking is allowed, the Dynamic Programming approach cannot be applied in general. The reason for this will be explained in Section B-2. In finding an optimum solution the Branch and Bound technique is used. A simple example is presented at the end of Section B-3, in which an optimal solution is obtained by using the Branch and Bound technique.

2. 1 part and M machine centers without backtracking

The total cost of the production system is the total setup cost plus the total inventory holding cost. The setup cost of machine center i per year is $S_i \cdot N_i$ and its total is $\sum_{i=1}^{M} S_i \cdot N_i$. By combining this with equation 3.9, the total cost can be expressed as follows:

$$
\begin{aligned}
(\text{The total production cost per year}) = {} & S_1 \cdot N_1 + \frac{I.V_1}{2} \frac{D}{N_1}(1 - \frac{D}{P_1}) \\
& + \sum_{i=2}^{M} \{S_i N_i + \frac{IV_i D}{2N_i} (1 - \frac{D}{P_i}) \\
& + \frac{IV_{i-1}}{2} \cdot \frac{D}{N_i} [(1 - \frac{D}{P_i})(1 - \frac{1}{\alpha_{i-1}}) \\
& + \frac{2}{k_{i-1}} \frac{D}{\alpha_{i-1}} \frac{1}{P_{i-1}} \\
& + (\frac{k_{i-1}-1}{k_{i-1}}) \frac{D}{\alpha_{i-1}} (\frac{1}{P_{i-1}} - \frac{1}{P_i}) \delta(P_i - P_{i-1})]\} \\
& + \frac{D}{N_M} \cdot \frac{D}{n_M} \cdot \frac{1}{P_M} \cdot V_M \cdot I
\end{aligned}
\tag{3.10}
$$

In equation 3.10 all k_i's $(i=1,2,\ldots,M-1)$ and n_M are given constants. In particular $k_i = \frac{n_i}{\beta_{i+1}}$ and a positive integer. α_i and β_{i+1} are relatively prime integers where $\frac{N_{i+1}}{N_i} = \frac{\beta_{i+1}}{\alpha_i}$.

An optimum number of cycles for each machine center can be obtained by applying the Dynamic Programming Algorithm to equation 3.10.

Let STERM(N_i) represent the summation of all the terms in equation 3.10, which contain the variable N_i. For example, STERM(N_1) = $S_1 N_1 + \frac{IV_1}{2} \frac{D}{N_1} (1 - \frac{D}{P_1})$. Given N_n and N_{n-1}, let $f_n(N_n, N_{n-1})$ =

$\text{STERM}(N_n) + f^*_{n-1}(N_{n-1})$ where $f^*_{n-1}(N_{n-1}) = \min_{N_1,\ldots,N_{n-2}} \{ \sum_{i=1}^{n-1} \text{STERM}(N_i)\}$.

For a given N_n, let $f_n(N_n, N^*_{n-1})$ be the $\min_{N_{n-1}} \{f_n(N_n, N_{n-1})\}$. Then $f_n(N_n, N^*_{n-1}) = f^*_n(N_n)$. From the recurrence relationship, $f_n(N_n, N_{n-1}) = \text{STERM}(N_n) + f^*_{n-1}(N_{n-1})$, it is possible to find $f^*_M(N_M)$ for all possible values of N_M. The resultant optimum total production cost is $\min_{N_M} \{f^*_M(N_M)\}$ and the set of optimum cycles, $N_1, N_2, \ldots, N_M$, can be found very easily once $\min_{N_M} \{f^*_M(N_M)\}$ has been obtained.

The direction of this algorithm is forward. However, it is possible to execute the algorithm backward by modifying the recurrence relationship.

3. N parts and M machine centers with backtracking

When there is more than one different part or backtracking is allowed, the Dynamic Programming Algorithm cannot be applied to the case in general. The main reason for this is the fact that the recurrence relationship may not exist and the principle of optimality may not hold. When there is only one part and backtracking is not allowed, the decisions made on machine centers 1 through n-2 do not affect the value of $STERM(N_n)$. On the other hand, the value of $f^*_{n-1}(N_{n-1})$ will not be affected by the values of $N_n, N_{n+1}, \ldots, N_M$.

These facts guarantee the existence of the recurrence relationship and the realization of the principle of optimality. For a multi-part or backtracking case, the operation sequences can nullify these facts and make the application of the Dynamic Programming impossible. However, the set of optimum cycles can be found by using the Branch and Bound technique. Before applying this technique to the problem, it is necessary to develop a general expression for the total production cost. The following symbols are defined for the expression.

D_j: Yearly demand rate for part j. $j=1,2,\ldots,N$

P_{kj}: The production rate of part j at its kth operation

V_{kj}: The value of part j just after it has finished its kth operation

S_{kj}: The setup cost per cycle of part j at its kth operation

N_{kj}: The number of cycles of the machine center which is doing the kth operation of part j

n_{kj}: The number of containers used to move part j from the machine center which is doing the kth operation of part j to the machine center which is doing its k+1th operation.

α_{kj} and $\beta_{k+1,j}$: Relatively prime integers such that

$$\frac{N_{k+1,j}}{N_{kj}} = \frac{\beta_{k+1,j}}{\alpha_{kj}}$$

k_{kj}: A predetermined positive integer such that

$$k_{kj} = \frac{n_{kj}}{\beta_{k+1,j}}$$

ℓ_j: The total number of different operations of part j

In the definition of the above symbols the subscript k represents the kth operation of the operation sequence of part j.

The portion of the total production cost per year to be charged to part j can be expressed as follows by substituting the subscript i with kj in equation 3.10.

(The production cost of part j per year)

$$
\begin{aligned}
&= S_{1j} \cdot N_{1j} + \frac{I \cdot V_{1j}}{2} \cdot \frac{D_j}{N_{1j}} (1 - \frac{D_j}{P_{1j}}) \\
&+ \sum_{k=2}^{\ell j} \{S_{kj} \cdot N_{kj} + \frac{I \cdot V_{kj}}{2} \cdot \frac{D_j}{N_{kj}} (1 - \frac{D_j}{P_{kj}}) \\
&+ \frac{I \cdot V_{k-1,j}}{2} \cdot \frac{D_j}{N_{kj}} [(1 - \frac{D_j}{P_{kj}})(1 - \frac{1}{\alpha_{k-1,j}}) \\
&+ \frac{2}{k_{k-1,j}} \cdot \frac{D_j}{\alpha_{k-1,j}} \cdot \frac{1}{P_{k-1,j}} \\
&+ (\frac{k_{k-1,j} - 1}{k_{k-1,j}}) \cdot \frac{D_j}{\alpha_{k-1,j}} (\frac{1}{P_{k-1,j}} - \frac{1}{P_{kj}}) \, \delta(P_{kj} - P_{k-1,j})]\} \\
&+ \frac{D_j}{N_{\ell_j,j}} \cdot \frac{D_j}{n_{\ell_j,j}} \cdot \frac{1}{P_{\ell_j,j}} \cdot V_{\ell_j,j} \cdot I \qquad (3.11)
\end{aligned}
$$

Equation 3.11 corresponds to the total cost of a production system consisting of one part, part j, and ℓ_j machine centers. The production cost for each part j can be obtained similarly by equation 3.11. The total production cost per year of this multi-part production system, is, then, simply the summation of all the production costs of N parts.

(Total production cost per year)

$$= \sum_{j=1}^{N} \text{(The production cost of part } j \text{ per year)} \tag{3.12}$$

In Section B-2, $\text{STERM}(N_i)$ is defined to be the summation of all the terms which include the variable N_i in equation 3.10. In this section, $\text{STERM}(N_i)$ is defined similarly. It represents the summation of all the terms which have the variable N_i in equation 3.12. The actual form of $\text{STERM}(N_i)$ depends on the operation sequences of the N parts. Nevertheless the general form of $\text{STERM}(N_i)$ is as follows:

$$\text{STERM}(N_i) = \sum_{j\varepsilon\Phi_i} \sum_{k\varepsilon\Psi_{ij}} [S_{kj} \cdot N_i$$

$$+ \frac{I \cdot V_{kj}}{2} \cdot \frac{D_j}{N_i} (1 - \frac{D_j}{P_{kj}})]$$

$$+ \sum_{j\varepsilon\Pi_i} \frac{D_j}{N_i} \cdot \frac{D_j}{n_{\ell_j,j}} \cdot \frac{1}{P_{\ell_j,j}} \cdot V_{\ell_j,j} \cdot I$$

$$+ \sum_{j\varepsilon\Phi_i} \sum_{\substack{k\varepsilon\Psi_{ij} \\ k\neq 1}} \frac{I \cdot V_{k-1,j}}{2} \cdot \frac{D_j}{N_i} [(1 - \frac{D_j}{P_{kj}})(1 - \frac{1}{\alpha_{k-1,j}})$$

$$+ \frac{2}{k_{k-1,j}} \frac{D_j}{\alpha_{k-1,j}} \frac{1}{P_{k-1,j}}$$

$$+ (\frac{k_{k-1,j} - 1}{k_{k-1,j}}) \frac{D_j}{\alpha_{k-1,j}} (\frac{1}{P_{k-1,j}} - \frac{1}{P_{kj}}) \delta(P_{kj} - P_{k-1,j})] \qquad (3.13)$$

In equation 3.13, Φ_i, Ψ_{ij} and Π_i represent sets of j's and k's. The following are the definitions of these sets.

Φ_i: The set of parts which visit machine center i at least once

Ψ_{ij}: The operations of part j, which are performed at machine center i

Π_i: The set of parts which visit machine center i for its last operation

Equation 3.13 will be acquired by writing down equation 3.11 for each part and then collecting and adding all the terms which have N_i in their expressions. Equation 3.12 can be rewritten using the term, STERM(N_i).

(Total production cost per year)

$$= \sum_{i=1}^{M} \text{STERM}(N_i) \qquad (3.14)$$

In order to apply the Branch and Bound technique it is necessary to acquire a tight lower bound for a given branch. This is possible via equation 3.13. The last term in equation 3.13 includes $\alpha_{k-1,j}$ in three places. By factoring out $\alpha_{k-1,j}$ the last term can be rearranged as follows:

$$\sum_{j\epsilon\Phi_i} \sum_{\substack{k\epsilon\Psi_{ij} \\ k\neq 1}} \frac{I \cdot V_{k-1,j}}{2} \cdot \frac{D_j}{N_i} \{(1 - \frac{D_j}{P_{kj}}) - \frac{1}{\alpha_{k-1,j}} [(1 - \frac{D_j}{P_{kj}})$$

$$- \frac{2}{k_{k-1,j}} \cdot \frac{D_j}{P_{k-1,j}} - (\frac{k_{k-1,j} - 1}{k_{k-1,j}})(\frac{D_j}{P_{k-1,j}} - \frac{D_j}{P_{kj}}) \delta(P_{kj} - P_{k-1,j})]\} .$$

The possible value of $\alpha_{k-1,j}$ ranges from 1 to the maximum value of $N_{k-1,j}$, i.e., $\bar{N}_{k-1,j}$. Assume the value inside the bracket [.] is positive for given j and k. Then the value inside the bracket {.} will be minimized by setting $\alpha_{k-1,j} = 1$. When the value of the bracket [.] is negative, the value of the bracket {.} will be minimized by setting $\alpha_{k-1,j} = \bar{N}_{k-1,j}$. For each possible value of N_i, this procedure will provide the minimum value of STERM(N_i). Let MSTERM(N_i) represent such minimum value of STERM(N_i).

The efficiency of this technique will depend on the value of the current upper bound. The smaller this value, the smaller the total number of branches to be examined. Since the rates of production are assumed to be much greater than the rates of demand, the value inside the bracket {.} will be minimized in general when

$\alpha_{k-1,j} = 1$. This justifies choosing the current upper bound among feasible solutions in which all the machine centers have one common cycle number. The current upper bound will be the minimum value among these feasible solutions. The common cycle is less than or equal to $\min(\bar{N}_1, \bar{N}_2, \ldots, \bar{N}_M)$.

The branching begins with N_1. Any branch whose lower bound exceeds the current upper bound will be excluded from further consideration. When the value of a feasible solution is found to be smaller than the current upper bound during the branching operation, that value will be the new current upper bound.

An example is appropriate at this point. The meanings and definitions of the symbols of equations 3.11 and 3.13 will be clarified via the example. Also the step-by-step procedures in applying the technique to the example are to be shown.

[Example 1.]: A production system is producing 3 different parts. There are 3 machine centers. The operation sequence of each of the parts is shown below.

Part 1: (1, 2, 3, 1)

Part 2: (3, 2, 1)

Part 3: (2, 1, 2, 3)

The following are the data for this production system.

a) Maximum cycles/yr: $\bar{N}_1 = 4$, $\bar{N}_2 = 6$, $\bar{N}_3 = 7$

b) Demand rate/yr: $D_1 = 6{,}000$, $D_2 = 7{,}000$, $D_3 = 8{,}000$

c) Production rate/yr:

i) part 1: $P_{11} = 6 \times 10^4$, $P_{21} = 8 \times 10^4$,
$P_{31} = 15 \times 10^4$, $P_{41} = 1 \times 10^4$

ii) part 2: $P_{12} = 3 \times 10^4$, $P_{22} = 4 \times 10^4$,
$P_{32} = 3 \times 10^4$

iii) part 3: $P_{13} = 15 \times 10^4$, $P_{23} = 2 \times 10^5$,
$P_{33} = 12 \times 10^4$, $P_{43} = 7 \times 10^4$

d) k_{kj} and $n_{\ell_j,j}$: $(\ell_1 = 4,\ \ell_2 = 3,\ \ell_3 = 4)$

i) part 1: $k_{11} = 3$, $k_{21} = 4$, $k_{31} = 2$, $n_{41} = 20$

ii) part 2: $k_{12} = 5$, $k_{22} = 10$, $n_{32} = 35$

iii) part 3: $k_{13} = 6$, $k_{23} = 7$, $k_{33} = 5$, $n_{43} = 20$

e) Unit cost of parts (V_{kj}): \$/unit

i) part 1: $V_{11} = 5$, $V_{21} = 7$, $V_{31} = 10$, $V_{41} = 12$

ii) part 2: $V_{12} = 30$, $V_{22} = 35$, $V_{32} = 40$

iii) part 3: $V_{13} = 3$, $V_{23} = 10$, $V_{33} = 20$, $V_{43} = 50$

f) Setup costs (S_{kj}): \$/setup

i) part 1: $S_{11} = 2000$, $S_{21} = 1500$, $S_{31} = 700$,

$$S_{41} = 1000$$

ii) part 2: $S_{12} = 200$, $S_{22} = 200$, $S_{32} = 1200$,

iii) part 3: $S_{13} = 100$, $S_{23} = 800$, $S_{33} = 200$,

$$S_{43} = 400$$

g) Yearly interest rate: $I = 0.2$

(Solution Procedures):

Step 1). Determine the 3 sets defined in equation 3.13.

Φ_i: $\Phi_1 = (1,2,3)$, $\Phi_2 = (1,2,3)$, $\Phi_3 = (1,2,3)$

Ψ_{ij}: $\Psi_{11} = (1,4)$, $\Psi_{12} = (3)$, $\Psi_{13} = (2)$

$\Psi_{21} = (2)$, $\Psi_{22} = (2)$, $\Psi_{23} = (1,3)$

$\Psi_{31} = (3)$, $\Psi_{32} = (1)$, $\Psi_{33} = (4)$

Π_i: $\Pi_1 = (1,2)$, $\Pi_3 = (3)$

Step 2). Express STERM(N_i) in terms of the symbols which represent the data for each N_i. In this example only STERM(N_1) is shown. STERM(N_2) and STERM(N_3) can be expressed similarly. Before expressing STERM(N_1), writing down the sets under the Σ sign is helpful.

$$\Phi_1 = \begin{pmatrix} 1 & , & 2 & , & 3 \\ \downarrow & & \downarrow & & \downarrow \\ \Psi_{11} \doteq (1,4) & & \Psi_{12} \doteq (3) & & \Psi_{13} \doteq (2) \end{pmatrix}$$

$$\Pi_1 = (1, 2)$$

$$\Phi_1 = \begin{pmatrix} 1 & , & 2 & , & 3 \\ \downarrow & & \downarrow & & \downarrow \\ \Psi_{11,\,k\neq 1} \doteq (4) & & \Psi_{12,\,k\neq 1} \doteq (3) & & \Psi_{13,\,k\neq 1} \doteq (2) \end{pmatrix}$$

$$\begin{aligned}
\mathrm{STERM}(N_1) = {} & [S_{11} \cdot N_1 + \frac{I \cdot V_{11}}{2} \cdot \frac{D_1}{N_1} (1 - \frac{D_1}{P_{11}})] \\
& + [S_{41} \cdot N_1 + \frac{I \cdot V_{41}}{2} \cdot \frac{D_1}{N_1} (1 - \frac{D_1}{P_{41}})] \\
& + [S_{32} \cdot N_1 + \frac{I \cdot V_{32}}{2} \cdot \frac{D_2}{N_1} (1 - \frac{D_2}{P_{32}})] \\
& + [S_{23} \cdot N_1 + \frac{I \cdot V_{23}}{2} \cdot \frac{D_3}{N_1} (1 - \frac{D_3}{P_{23}})] \\
& + \frac{D_1}{N_1} \cdot \frac{D_1}{n_{41}} \cdot \frac{1}{P_{41}} \cdot V_{41} \cdot I \\
& + \frac{D_2}{N_1} \cdot \frac{D_2}{n_{32}} \cdot \frac{1}{P_{32}} \cdot V_{32} \cdot I \\
& + \frac{I \cdot V_{31}}{2} \cdot \frac{D_1}{N_1} [(1 - \frac{D_1}{P_{41}})(1 - \frac{1}{\alpha_{31}}) \\
& + \frac{2}{k_{31}} \cdot \frac{D_1}{\alpha_{31}} \cdot \frac{1}{P_{31}} \\
& + (\frac{k_{31} - 1}{k_{31}}) \frac{D_1}{\alpha_{31}} (\frac{1}{P_{31}} - \frac{1}{P_{41}}) \delta(P_{41} - P_{31})]
\end{aligned}$$

$$+ \frac{I \cdot V_{22}}{2} \cdot \frac{D_2}{N_1} [(1 - \frac{D_2}{P_{32}})(1 - \frac{1}{\alpha_{22}})$$

$$+ \frac{2}{k_{22}} \cdot \frac{D_2}{\alpha_{22}} \cdot \frac{1}{P_{22}}$$

$$+ (\frac{k_{22} - 1}{k_{22}}) \frac{D_2}{\alpha_{22}} (\frac{1}{P_{22}} - \frac{1}{P_{32}}) \delta(P_{32} - P_{22})]$$

$$+ \frac{I \cdot V_{13}}{2} \cdot \frac{D_3}{N_1} [(1 - \frac{D_3}{P_{23}})(1 - \frac{1}{\alpha_{13}})$$

$$+ \frac{2}{k_{13}} \cdot \frac{D_3}{\alpha_{13}} \cdot \frac{1}{P_{13}}$$

$$+ (\frac{k_{13} - 1}{k_{13}}) \frac{D_3}{\alpha_{13}} (\frac{1}{P_{13}} - \frac{1}{P_{23}}) \delta(P_{23} - P_{13})].$$

STERM(N_2) and STERM(N_3) can be expressed in similar fashion.

Step 3). Obtain MSTERM(N_1), which is the minimum value of STERM(N_1), for each possible value of N_1. Do the same for STERM(N_2) and STERM(N_3). To obtain MSTERM(N_1), it is necessary to check the signs of the following terms and set the values of α_{31}, α_{22}, and α_{13} equal to 1 or limit values.

$$\text{(a):} \quad [(1 - \frac{D_1}{P_{41}}) - \frac{2}{k_{31}} \cdot \frac{D_1}{P_{31}}$$

$$- (\frac{k_{31} - 1}{k_{31}})(\frac{D_1}{P_{31}} - \frac{D_1}{P_{41}}) \delta(P_{41} - P_{31})]$$

$$(b): \quad [(1 - \frac{D_2}{P_{32}}) - \frac{2}{k_{22}} \cdot \frac{D_2}{P_{22}}$$
$$- (\frac{k_{22} - 1}{k_{22}}) (\frac{D_2}{P_{22}} - \frac{D_2}{P_{32}}) \ \delta(P_{32} - P_{22})]$$

$$(c): \quad [(1 - \frac{D_3}{P_{23}}) - \frac{2}{k_{13}} \cdot \frac{D_3}{P_{13}}$$
$$- (\frac{k_{13} - 1}{k_{13}})(\frac{D_3}{P_{13}} - \frac{D_3}{P_{23}}) \ \delta(P_{23} - P_{13})]$$

If (a) is positive, set $\alpha_{31} = 1$. If (a) is negative, set $\alpha_{31} = \bar{\bar{N}}_{31} = \bar{\bar{N}}_3 = 7$. $\bar{\bar{N}}_{31}$ represents the maximum cycles per year for the machine center which is performing the third operation of part 1. This machine center is machine center 3 and its maximum cycles per year are 7. The checking of the signs and assignment of values to α_{22} and α_{13} for (b) and (c) will be similar. By substituting the given data into (a), (b) and (c), the following are obtained.

$$(a): \quad \frac{54}{150} \rightarrow \alpha_{31} = 1$$

$$(b): \quad \frac{439}{600} \rightarrow \alpha_{22} = 1$$

$$(c): \quad \frac{419}{450} \rightarrow \alpha_{13} = 1$$

The minimum values of $STERM(N_1)$ for $N_1 = 1,2,3,4$ are obtained by substituting all the pertinent data into the expression in step 2 with $\alpha_{31} = \alpha_{22} = \alpha_{13} = 1$. The minimum values of STERM(N_2) and STERM(N_3) are obtained similarly.

Step 4). Prepare a table showing the computed values of MSTERM(N_i) in step 3.

Table 3.1. Calculated values of MSTERM(N_i)

Values of MSTERM(N_i) \ N_i	1	2	3	4	5	6	7
MSTERM(N_1)	41379	28189	27126	29095			
MSTERM(N_2)	48034	25517	18678	15759	14407	13839	
MSTERM(N_3)	59740	31820	23380	19810	18188	17540	17449

Step 5). Obtain the current upper bound from the feasible solutions which have a common number of cycles for all the machine centers.

$N_1 = N_2 = N_3 = 1$: Total production cost/yr = 149153

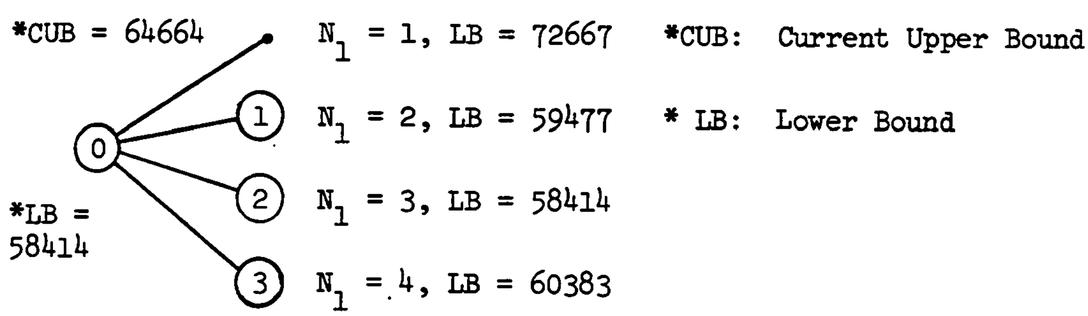
$N_1 = N_2 = N_3 = 2$: Total production cost/yr = 85526

$N_1 = N_2 = N_3 = 3$: Total production cost/yr = 69184

$N_1 = N_2 = N_3 = 4$: Total production cost/yr = 64664

The minimum value of these feasible solutions is 64,664 and this is the current upper bound. The values of the feasible solutions happen to be the sums of each column of the table in step 4. This will be always the case as long as all the α's are set equal to 1 in step 3.

Step 6). Proceed the Branch and Bound Algorithm beginning with N_1. Node 0 is the starting point. Its lower bound is min{MSTERM(N_1)} + min{MSTERM(N_2)} + min{MSTERM(N_3)} = 27126 + 13839 + 17449 = 58414. This is the overall lower bound. Four branches are coming out from node 0, and their lower bounds are calculated as follows: LB = 58414 - min{MSTERM(N_1)} + MSTERM(N_1). Since N_1 takes a specific value for each of the four branches, min{MSTERM(N_1)} should be replaced with MSTERM(N_1) in calculating the lower bounds. The following shows the lower bounds of the four branches from node 0.

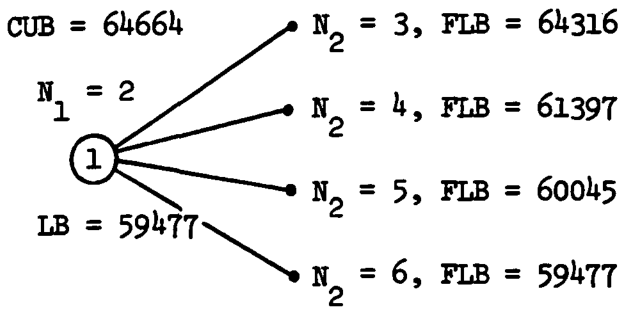


Branches which have lower bounds bigger than CUB will be excluded from further consideration. N_1 = 1 is one such branch.

One point to mention is that MSTERM(N_i) is a U-shaped function of N_i. Since the lower bounds of the above branches are 58414 - min{MSTERM(N_1)} + MSTERM(N_1), it is preferable to start the calculation at the minimum point of MSTERM(N_1), i.e., N_1 = 3. Branching and the calculation for its lower bound will go on to the left of this point as long as $\text{MSTERM}(N_1) \le \text{CUB} - 58414 + \min\{\text{MSTERM}(N_1)\}$ $= 33376$. The same will apply to the right of the minimum point. This procedure will speed up the overall efficiency of the algorithm when $\bar{N}_1$ is a large number and procedure will be used for every branching operation to increase the efficiency of the algorithm. If this procedure were used, the branch $N_1 = 1$ would not be considered.

There are three branches left for further consideration. From node 1, the potential number of branches is 6 as N_2 assumes 1, 2, ..., 6. Two types of lower bound will be calculated for them. The first type which is called the First Lower Bound, is to be obtained from the LB of node 1 and the table in step 4. The second type, which is called the Second Lower Bound, is to be obtained from the First Lower Bound and the expression of $\text{STERM}(N_i)$ in step 2. The value of the SLB, the Second Lower Bound, will be always bigger than or equal to the value of the FLB, the First Lower Bound, for a given node. The FLB is used for a quick initial elimination of any unpromising branches. The SLB is used for further elimination.

The LB of node 1 is $58414 - \min\{\text{MSTERM}(N_1)\} + \text{MSTERM}(N_1 = 2)$ $= \min\{\text{MSTERM}(N_2)\} + \min\{\text{MSTERM}(N_3)\} + \text{MSTERM}(N_1 = 2) = 59477$. Since N_2 will take a specific value for each of the new branches from node 1, $\min\{\text{MSTERM}(N_2)\}$ should be replaced with $\text{MSTERM}(N_2)$ in calculating their lower bounds. Then the FLB will be calculated as follows: $\text{FLB} = 59477 - \min\{\text{MSTERM}(N_2)\}$ $+ \text{MSTERM}(N_2)$. Since $\text{CUB} - 59477 + \min\{\text{MSTERM}(N_2)\} = 19026$, the branches of $N_2 = 1$ and $N_2 = 2$ will be eliminated. The FLB's of the other branches are shown below.



When $N_1 = 2$ and N_2 takes one of (3,4,5,6), the values of those α's which are related to both N_1 and N_2 may not be 1. If some of them are no longer 1, it is possible to obtain another lower bound which is much bigger than the FLB for some of the above four branches. This new lower bound is the SLB. As pointed out before, this will be obtained from the FLB and the expression of STERM(N_i) in step 2.

Before calculating for the SLB's, writing down the terms which include the α's in the expression of STERM(N_i) according to the following fashion will save time and effort.

STERM(N_1)	MSTERM(N_1)	α's
$\frac{1}{N_1} \cdot 6000 \cdot \{\frac{4}{10} - \frac{1}{\alpha_{31}} \cdot \frac{54}{150}\}$,	$\frac{1}{N_1} \cdot 240$,	$\alpha_{31} \to \frac{N_1}{N_3}$
$\frac{1}{N_1} \cdot 24500 \cdot \{\frac{23}{30} - \frac{1}{\alpha_{22}} \cdot \frac{439}{600}\}$,	$\frac{1}{N_1} \cdot 857.5$,	$\alpha_{22} \to \frac{N_1}{N_2}$
$\frac{1}{N_1} \cdot 2400 \cdot \{\frac{24}{25} - \frac{1}{\alpha_{13}} \cdot \frac{419}{450}\}$,	$\frac{1}{N_1} \cdot 69.3$,	$\alpha_{13} \to \frac{N_1}{N_2}$

STERM(N_2)	MSTERM(N_2)	α's
$\frac{1}{N_2} \cdot 3000 \cdot \{\frac{74}{80} - \frac{1}{\alpha_{11}} \cdot \frac{101}{120}\}$,	$\frac{1}{N_2} \cdot 250$,	$\alpha_{11} \to \frac{N_2}{N_1}$
$\frac{1}{N_2} \cdot 21000 \cdot \{\frac{33}{40} - \frac{1}{\alpha_{12}} \cdot \frac{411}{600}\}$,	$\frac{1}{N_2} \cdot 2940$,	$\alpha_{12} \to \frac{N_2}{N_3}$
$\frac{1}{N_2} \cdot 8000 \cdot \{\frac{112}{120} - \frac{1}{\alpha_{23}} \cdot \frac{484}{525}\}$,	$\frac{1}{N_2} \cdot 91.43$,	$\alpha_{23} \to \frac{N_2}{N_1}$

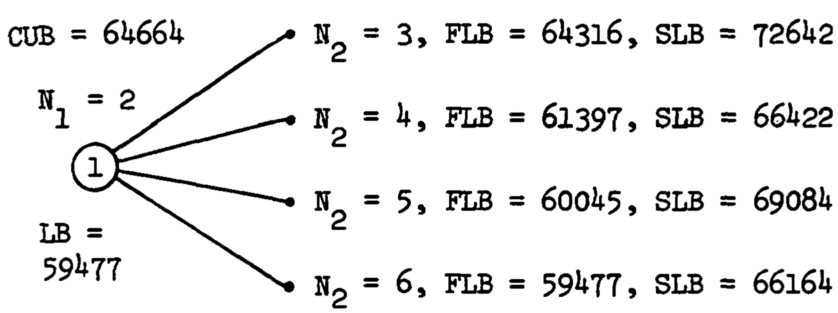
STERM(N_3)	MSTERM(N_3)	α's
$\frac{1}{N_3} \cdot 4200 \cdot \{\frac{144}{150} - \frac{1}{\alpha_{21}} \cdot \frac{717}{800}\}$,	$\frac{1}{N_3} \cdot 267.75$,	$\alpha_{21} \to \frac{N_3}{N_2}$
$\frac{1}{N_3} \cdot 16000 \cdot \{\frac{62}{70} - \frac{1}{\alpha_{33}} \cdot \frac{451}{525}\}$,	$\frac{1}{N_3} \cdot 426.67$,	$\alpha_{33} \to \frac{N_3}{N_2}$

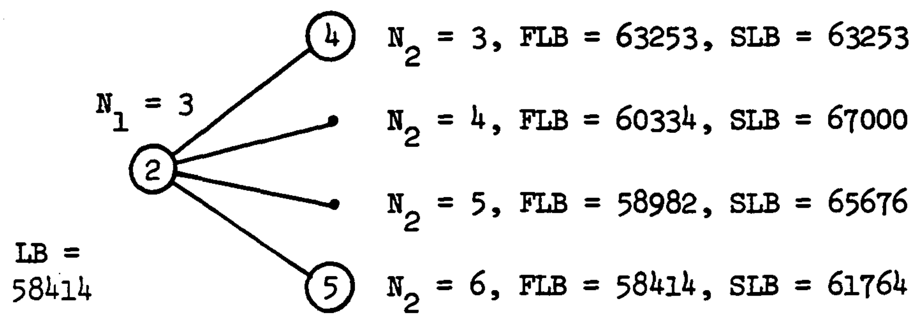
The terms in the first column are those containing α's in the expression STERM(N_i). The same terms appear in the second column where the α's take values of 1 or the limit values. These terms appear in the calculation for MSTERM(N_i). The third column shows the relationships between the α's and N_i. For example, α_{31} and β_{41} are relatively prime integers where $\frac{N_1}{N_3} = \frac{\beta_{41}}{\alpha_{31}}$.

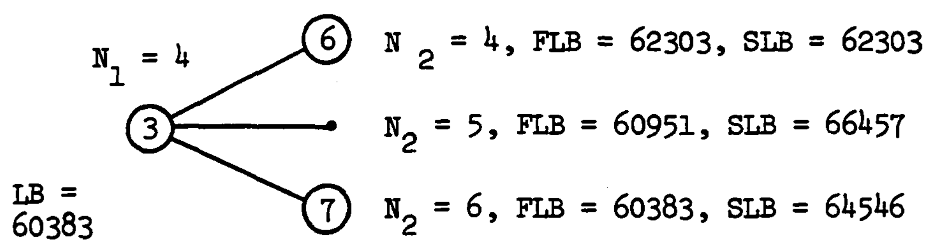
When $N_1 = 2$ and $N_2 = 3$, the values of the α's will change as follows: $\alpha_{31} = 1$, $\alpha_{22} = \alpha_{13} = 3$, $\alpha_{11} = \alpha_{23} = 2$, $\alpha_{12} = 1$, and $\alpha_{21} = \alpha_{23} = 1$. It should be noted that the α's which do not relate to both N_1 and N_2 are allowed to assume 1. Since the terms in the first column are the only terms to be affected by the new values of the α's, the SLB of this branch can be calculated as follows:

$$\begin{aligned} SLB(N_1 = 2, N_2 = 3) = 64316 &+ \frac{1}{2} \cdot 245000 \cdot \{\frac{23}{30} - \frac{1}{3} \cdot \frac{439}{600}\} \\ &- \frac{1}{2} \cdot 857.5 \\ &+ \frac{1}{2} \cdot 2400 \cdot \{\frac{24}{25} - \frac{1}{3} \cdot \frac{419}{450}\} - \frac{1}{2} \cdot 69.3 \\ &+ \frac{1}{3} \cdot 3000 \cdot \{\frac{74}{80} - \frac{1}{2} \cdot \frac{101}{120}\} - \frac{1}{3} \cdot 250 \\ &+ \frac{1}{3} \cdot 8000 \cdot \{\frac{112}{120} - \frac{1}{2} \cdot \frac{484}{525}\} - \frac{1}{3} \cdot 91.43 = 72642 \end{aligned}$$

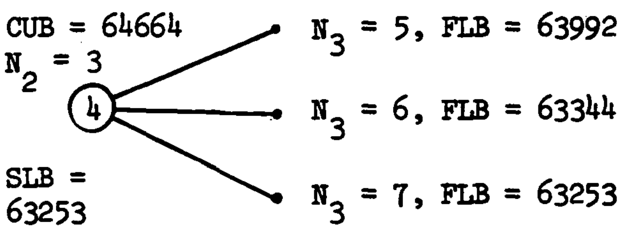
Since $SLB(N_1 = 2, N_2 = 3) > CUB$, this branch will be excluded. Similar computations will disclose the SLB's of all the other branches from nodes 1, 2, and 3. They are shown below.







Nodes 4, 5, 6 and 7 are the only branches which have survived the elimination by the SLB's. The branching operation will continue beginning with node 4. The total number of the potential branches from node 4 is 7 as N_3 assumes 1, 2, ..., 7. As before, their FLB's are to be computed first. The FLB of node 4 is MSTERM(N_1 = 3) + MSTERM(N_2 = 3) + min{MSTERM(N_3)} = 63253. Its SLB is STERM(N_1 = 3) + STERM(N_2 = 3) + min{MSTERM(N_3)} = 63253 where the values of all α's are 1. The fact that all α's = 1 in the calculation for the SLB is the reason for the equality between the two lower bounds. Since N_3 assumes a specific value for each of the 7 branches, the FLB's are calculated as follows: FLB = 63253 - min{MSTERM(N_3)} + MSTERM(N_3). Since CUB - 63253 + min{MSTERM(N_3)} = 18860, the branches of $N_3 = 1, 2, 3,$ and 4 will be eliminated. The FLB's of the remaining branches are shown below.



When $N_1 = 3$, $N_2 = 3$, and N_3 takes one of the set $(5, 6, 7)$, the values of the α's which are related to both N_1 and N_3 or N_2 and N_3 may take values other than 1. These are α_{31}, α_{12}, α_{21} and α_{33}. To be specific, when $N_1 = 3$, $N_2 = 3$, and $N_3 = 5$, $\alpha_{31} = 5$, $\alpha_{12} = 5$, and $\alpha_{21} = \alpha_{33} = 3$. By taking into account these changes, the SLB of this branch can be calculated as follows:

$$\mathrm{SLB}(N_1 = 3, N_i = 3, N_3 = 5) = 63992$$

$$+ \frac{1}{3} \cdot 6000 \cdot \{\frac{4}{10} - \frac{1}{5} \cdot \frac{54}{150}\} - \frac{1}{3} \cdot 240$$

$$+ \frac{1}{3} \cdot 21000 \cdot \{\frac{33}{40} - \frac{1}{5} \cdot \frac{411}{600}\} - \frac{1}{3} \cdot 2940$$

$$+ \frac{1}{5} \cdot 4200 \{\frac{144}{150} - \frac{1}{3} \cdot \frac{717}{800}\} - \frac{1}{5} \cdot 267.75$$

$$+ \frac{1}{5} \cdot 16000 \cdot \{\frac{62}{70} - \frac{1}{3} \cdot \frac{451}{525}\} - \frac{1}{5} \cdot 426.67 = 70758$$

Since $\mathrm{SLB}(N_1 = 3, N_2 = 3, N_2 = 5) > \mathrm{CUB}$, this branch will be eliminated. Similar calculations will disclose the SLB's of all the other branches from nodes 4, 5, 6 and 7. They are shown below.

CUB = 64664
$N_2 = 3$
(4)
SLB = 63253

- $N_3 = 5$, FLB = 63992, SLB = 70758
- $N_3 = 6$, FLB = 63344, SLB = 66084
- $N_3 = 7$, FLB = 63253, SLB = 69671

$N_2 = 6$
(5)
SLB = 61764

- $N_3 = 4$, FLB = 64125, SLB = 67981
- $N_3 = 5$, FLB = 62503, SLB = 67918
- (8) $N_3 = 6$, FLB = 61855, SLB = 62215 (New Current Upper Bound)
- $N_3 = 7$, FLB = 61764, SLB = 66527

$N_2 = 4$
(6)
SLB = 62303

- $N_3 = 5$, FLB = 63042, SLB = 68991
- $N_3 = 6$, FLB = 62394, SLB = 66637
- $N_3 = 7$, FLB = 62303, SLB = 70850

(7)
SLB = 64546

- $N_3 = 6$, FLB = 64637, SLB = 64997
- $N_3 = 7$, FLB = 64546, SLB = 69154

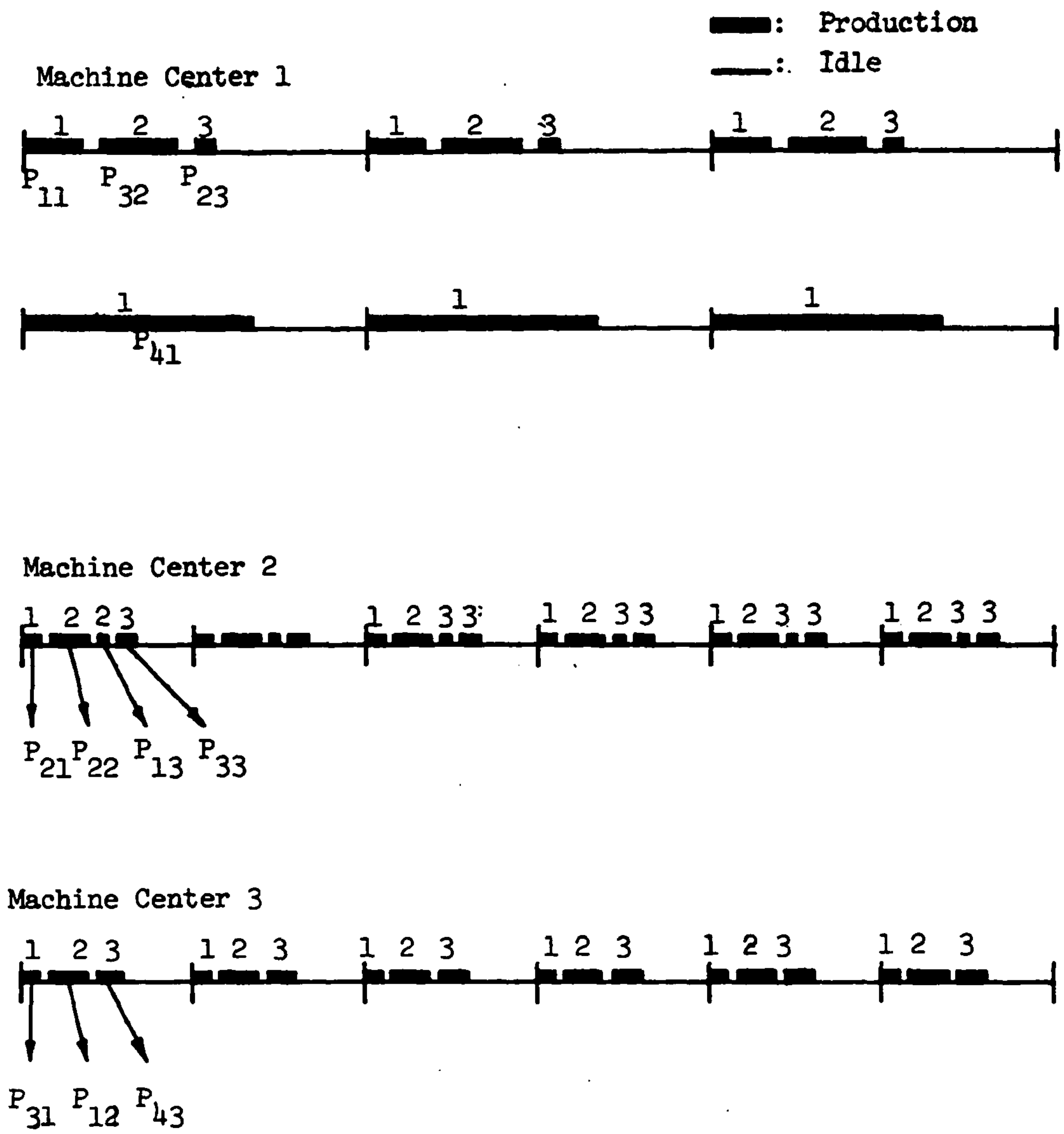
The SLB of node 8 is less than the CUB. Also it is a feasible solution. Hence its value, 62215, becomes the new current upper bound. Since there are no branches to be considered further, node 8 is the optimum solution. The optimum number of cycles per year of each machine center and its total production cost per year are as follows:

Machine Center	Optimum Cycles/Yr	Total Production Cost/Yr
1	3	
2	6	\$62,215
3	6	

The number of containers to be used at each operation of each part can be obtained by the relationship, $n_{kj} = k_{kj} \cdot \beta_{k+1,j}$.

Parts	1st Operation	2nd Operation	3rd Operation	4th Operation
1	6	4	2	20
2	5	10	35	
3	6	14	5	20

Step 7). Setup actual production scheduling. One important point to remember is to make sure that the sequences of each cycle are identical for a given machine center. There are many different ways to schedule this production system. The following is one example.



The numbers represent part numbers to be processed with rates of production specified for the scheduled production period. It happens that machine center 1 requires two identical machines. The total production cost per year, $62,215, will be realized if this sequence is to continue over a long period of time.

C. Optimum Production Capacities of Machine Centers with Non-Deteriorating Machine Capacities

1. Introduction

Production capacity or machine capacity is defined to be the available machine hours per year. In previous sections it was assumed that the available capacity of each machine center is infinite. Any additional machine capacity, if necessary, can be acquired without incurring cost. The total production cost, accordingly, does not include the cost which is associated with machine capacity.

In this section the infinite capacity assumption is relaxed and total production cost includes capacity cost. However, three restrictive assumptions are made. They are: 1) a known fixed cost occurs each year for carrying each identical machine at a given machine center which does not change over time, 2) the available machine hours of each machine do not decrease over time, and 3) there will be no machine replacement. The fixed cost includes maintenance cost, taxes, insurance, space cost, interest on the investment in the machines, etc. It should be noted that machine depreciation will not be included in the fixed carrying cost due to the third assumption. The problem is to find the optimum number of identical machines as well as the optimum production cycle for each machine center.

Even though a fixed cost for carrying individual machines is recognized, the situation is still far from reality. A machine involved in a production process always deteriorates physically in time as it is used and so does its production capacity. Its current market value usually goes down as it gets older and it becomes increasingly obsolete as technological innovations and breakthroughs are realized. Its maintenance cost increases every year. These factors make its replacement at some point of time inevitable. From this practical point of view, the three assumptions are highly improbable. However, this section has been inserted here as an extension of Section B as well as an intermediate step toward a more realistic model. In later sections all the three assumptions are relaxed and more realistic models are presented.

In Section B no setup time is considered explicitly even though the setup cost in recognized. In this section it is assumed that a fixed amount of production capacity is consumed for setup during each cycle for each operation on each part. If there were no setup time, the number of identical machines required at each machine center would not change no matter what production cycle is assigned. It is the setup time which links the decision on cycles to the decision on the number of machines.

When there is only one part, the number of identical machines required at each machine center is 1 disregarding their cycles. This is due to assumptions 16 and 17. In other words, the total machine hours required by a single part at any machine center should not be greater than the available machine hours of a single machine. Otherwise assumptions 16 and 17 would be violated. Because of these restrictions, the solution of the optimum number of machines for each machine center for the first case of Section B is trivial. The case to be dealt with is the second one where there are M machine centers and N parts with backtracking. Finding an optimum solution is essentially the same as before.

2. N parts and M machine centers with backtracking

The following definition of symbols is made in order to express the total production cost as well as related constraints.

$m_i(N_i)$: The minimum number of identical machines required at machine center i when its cycle is N_i per year

f_i: The fixed cost per year for carrying each identical machine at machine center i

t_{kj}: The setup time per cycle at kth operation of part j

It is easy to find $m_i(N_i)$ when the number of parts being processed at machine center i is small. As the number of parts visiting machine center i gets bigger, it might take some time to find $m_i(N_i)$. It should be noted that $m_i(N_i)$ is not another independent variable but a dependent variable of N_i.

With these symbols the total production cost per year and related constraints are as follows:

(Total production cost per year)

$$= \sum_{i=1}^{M} \{\mathrm{STERM}(N_i) + m_i(N_i)\ f_i\} \tag{3.15}$$

subject to

$$\bar{N}_i \cdot t_{kj} + \frac{D_j}{P_{kj}} \leq 1 \text{ for all } i\text{, where } i = 1, 2, \ldots, M$$

and $j \varepsilon \Phi_i$ and $k \varepsilon \Psi_{ij}$.

The detailed procedures in executing the Branch and Bound technique for an optimum solution are similar to those of Example 1 in Section B-3. The term $m_i(N_i) \cdot f_i$, capacity cost, causes some minor modifications. First, all the $STERM(N_i)$'s in the solution procedures of Example 1 are replaced with $STERM(N_i) + m_i(N_i) \cdot f_i$. Also all the $MSTERM(N_i)$'s are replaced with $MSTERM(N_i) + m_i(N_i) \cdot f_i$. The entries of the table in step 4 are, then, modified as follows:

Table 3.2. $MSTERM(N_i) + m_i(N_i) \cdot f_i$

$MSTERM(N_i) + m_i(N_i) \cdot f_i$ \ Values of N_i	1	2	. . .
$MSTERM(N_1) + m_1(N_1) \cdot f_1$	$MSTERM(N_1=1) + m_1(N_1=1) \cdot f_1$	$MSTERM(N_1=2) + m_1(N_1=2) \cdot f_1$	. . .
$MSTERM(N_2) + m_2(N_2) \cdot f_2$	$MSTERM(N_2=1) + m_2(N_2=1) \cdot f_2$	. .	. . .
.	.	.	. . .

There will be no changes for step 5 except that the values of feasible solutions are calculated by equation 3.15.

Because of the additional term $m_i(N_i) \cdot f_i$, $MSTERM(N_i) + m_i(N_i).f_i$ is not a U shaped function of N_i. Since $m_i(N_i)$ increases monotonically as N_i increases $MSTERM(N_i) + m_i(N_i) \cdot f_i$ will increase monotonically in the right side of the minimum point of MSTERM(N_i). But MSTERM(N_i) + $m_i(N_i) \cdot f_i$ may fluctuate in the left side of the minimum point of MSTERM(N_i). Accordingly, the branching and calculations for lower bounds will be performed for all possible cycles, i.e., 1, 2, ..., minimum point. To the left side of the minimum point the branching and calculation for lower bounds will continue as long as MSTERM(N_i) + $m_i(N_i) \cdot f_i \leq$ CUB − LB + min{MSTERM(N_i) + $m_i(N_i) \cdot f_i$}.

With these exceptions, the procedures are exactly the same as those of Example 1. The number of identical machines at each machine center is the additonal information provided by the optimum solution.

Even though the additional term, $m_i(N_i) \cdot f_i$, is added to MSTERM(N_i) and STERM(N_i), it will not increase the total number of branches to be considered. It simply adds one more term to each branch. The problem is to find $m_i(N_i)$ for each possible value of N_i at each machine center.

As defined before $m_i(N_i)$ represents the minimum number of identical machines required at machine center i when its production cycle is N_i per year. To find the minimum number of identical machines at a given machine center, each individual machine should be loaded as many parts as possible. As the number of parts increase at a given machine center, the total number of trials and errors may increase.

D. Optimum Machine Replacement Policy

1. Introduction

The three assumptions made in Section C, i.e., non-deteriorating machine capacities, the fixed cost for carrying individual machines, and no replacement, are relaxed in this section. The physical deterioration of machines is recognized by assuming a fixed amount of capacity decrease per year for each machine. The two standard assumptions of Terborgh (1949) are introduced to recognize the increasing maintenance cost and obsolescence. Replacement is signaled when the adverse minimum of defender is larger than that of challenger. One difference between Terborgh's model and this model is the treatment of the setup cost and WIP holding cost. While these costs are subsumed in the operating inferiority in Terborgh's model, they are recognized explicitly in this model. However, the recognition of these costs is achieved by making another simplifying assumption. The next section discusses more about the assumption.

Two replacement models are discussed in this section. The first one is the case where there are no budgeting constraints. The second one is the case where replacement decisions are made under some budgeting constraints. An example is given for the first model to show an actual application of the model.

2. A simplifying assumption

The last term of equation 3.13 represents the summation of the installation inventory holding costs of the machine centers which are immediately preceding machine center i in the production sequences of the parts being currently processed at machine center i. Its quantity is a function of N_i and the cycles of the preceding machine centers. If an assumption were made in such a way that it became a function of N_i only, equation 3.13 would be more manageable and easy to work with. Although this practical point of view is the prime motivation in making the assumption, there is some justification for the assumption also. If the number of different parts being processed at machine center i is large, their yearly demands are in similar magnitudes, and their unit values are comparable, then it is not unreasonable to replace the quantity in the last bracket, $[(1 - \frac{D_j}{P_{kj}})(1 - \frac{1}{\alpha_{k-1,j}})$

$$+ \frac{2}{k_{k-1,j}} \cdot \frac{D_j}{\alpha_{k-1,j}} \cdot \frac{1}{P_{k-1,j}}$$

$$+ (\frac{k_{k-1,j} - 1}{k_{k-1,j}}) \frac{D_j}{\alpha_{k-1,j}} (\frac{1}{P_{k-1,j}} - \frac{1}{P_{kj}}) \delta(P_{kj} - P_{k-1,j})],$$ with a

fixed constant τ_i. The introduction of τ_i may over estimate the installation inventory holding costs for some machine centers and under estimate that for some other machine centers. However, it is conjectured that the over estimation and the under estimation may cancel each other to a certain degree and the net result is a reasonable approximation of a correct figure.

The assignment of τ_i to the last bracket makes STERM(N_i) a function of N_i only. It also makes it possible to choose an optimum N_i disregarding the decisions of production cycles made on all the other machine centers. Above all, it is the simplifying assumption which makes it possible to recognize the setup cost and inventory holding cost explicitly in the calculation of the adverse minimum of challenger and defender. Under the assumption which has just been made, $STERMS(N_i)$ represents a modified form of $STERM(N_i)$ and is expressed as follows:

$$STERMS(N_i) = \sum_{j\epsilon\Phi_i} \sum_{k\epsilon\Psi_{ij}} [S_{kj} \cdot N_i + \frac{I \cdot V_{kj}}{2} \cdot \frac{D_j}{N_i} (1 - \frac{D_j}{P_{kj}})]$$

$$+ \sum_{j\epsilon\Pi_i} \frac{D_j}{N_i} \cdot \frac{D_j}{n_{\ell_j,j}} \cdot \frac{1}{P_{\ell_j,j}} \cdot V_{\ell_j,j} \cdot I$$

$$+ \sum_{j\epsilon\Phi_i} \sum_{\substack{k\epsilon\Psi_{ij} \\ k\neq 1}} \frac{I \cdot V_{k-1,j}}{2} \cdot \frac{D_j}{N_i} \cdot \tau_i \qquad (3.16)$$

3. Optimum production capacities of machine centers and replacement policy without budgeting constraints

Two situations are considered in this section. The first one is the case where the very beginning production capacity of machine center i is to be selected. The second one is the case where an optimum replacement decision is to be made at the beginning of a certain year.

Since there is no budgeting constraints, it is possible to replace any number of machines of machine center i as their adverse minimum is bigger than that of challenger. Accordingly, it is assumed that all the identical machines of machine center i are replaced at a same point of time. This means that all the machines will have a same age.

Adverse minimum is defined to be the lowest combined time-adjusted average of capital cost, operating inferiority, and STERMS(N_i). The capital cost is the repayment of and the return on the investments in the machines. The operating inferiority represents the difference between old machines and the best machines available in the market in times of their operating costs and service values. The difference originates from the deterioration and obsolescence of the old machines. STERMS(N_i) represents a portion of the total setup costs and WIP holding costs, which is directly related to N_i. It should be noted that the operating inferiority does not include the setup costs and WIP holding costs. The main part of it is maintenance costs.

The two standard assumptions of Terborgh's model are used in this model. The first one assumes that future challenger will have the same adverse minimum as the present one. The second one assumes that the present challenger will accumulate operating inferiority at a constant rate over its service life. While the challenger and the defender represent each single machine in Terborgh's model, they refer to each group of identical machines in this model.

d_i is defined to be a fixed amount of capacity decrease per year for each individual machine at machine center i. It is assumed that the decrement d_i occurs in lump sum at the beginning of each year disregarding the usage of each machine. It is also assumed that the decrement d_i is distributed evenly at the end of each cycle. When the machines are n years old, the available capacity is $1 - n \cdot d_i$.

As the available capacity of each machine decreases over time, it may be required to choose a smaller production cycle than the current one at the beginning of a certain year. This cycle change may cost a certain amount of money due to some necessary adjustments for WIP inventory. To keep the model simple it is assumed that the cost of cycle change is subsumed in the operating inferiority.

Figure 3.13 shows the relationship between STERMS(N_i) and N_i. N_i^* is a production cycle which minimizes STERMS(N_i). Note that STERMS(N_i) is a U-shaped function of N_i.

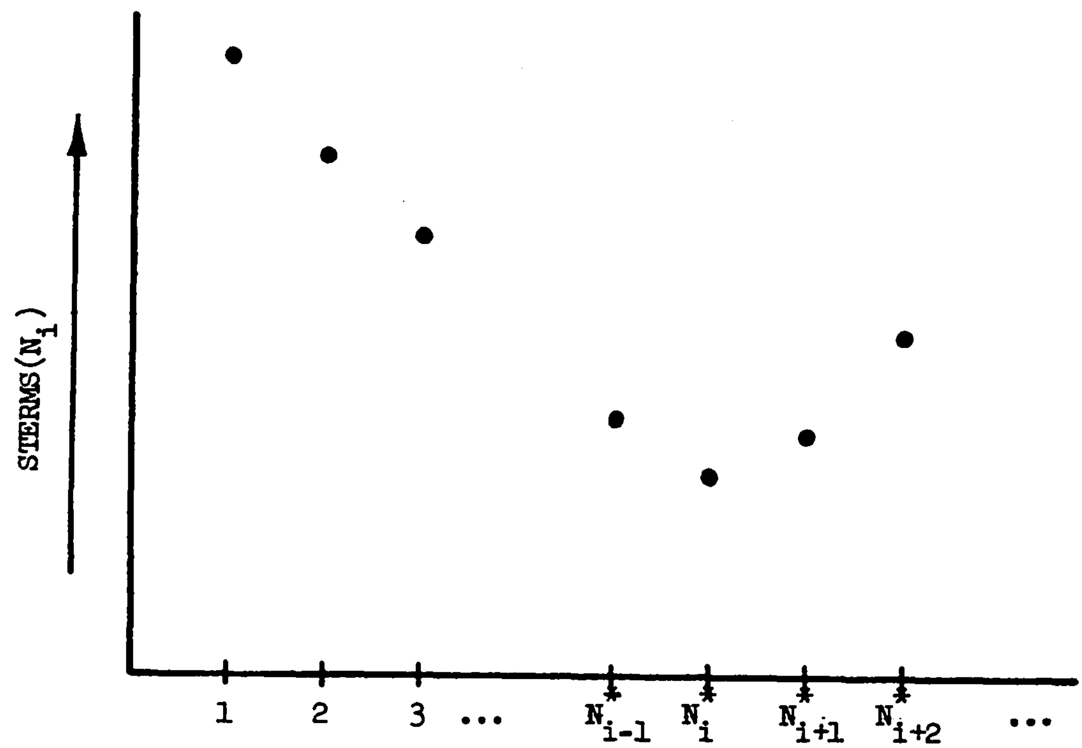


Figure 3.13. STERMS(N_i) and N_i

An optimum production cycle at a given point of time depends on the available capacity, N_i^* and $\bar{N}_i$. When there is sufficient capacity to keep $\min(N_i^*, \bar{N}_i)$, it is the optimum production cycle. Otherwise an optimum production cycle is the biggest cycle among the all possible cycles which are allowed by the current capacity. It is smaller than $\min(N_i^*, \bar{N}_i)$. With optimum production cycle, the value of $STERMS(N_i)$ is represented by $STERMS^*_{m_i,k}$ where m_i represents the number of machines at machine center i and k represents the age of the machines.

Suppose that machine center i starts with m_i brand new machines. All the parts to be processed at this machine center should be scheduled in such a way that the loading of each machine be well balanced. Otherwise the freedom in selecting a production cycle will be much more restricted. At the beginning of each year the production cycle of the previous year will be reviewed to decide whether it is still proper production cycle or not. If not, a new production cycle will be established, which is optimum with respect to the current capacity. In this case, it may be necessary to redistribute the work assignment to each machine for the sake of a balanced loading.

Since the available capacity of each machine decreases over time, there is a maximum life for the m_i machines. It is a time period beyond which the available capacity cannot satisfy the yearly demands of the parts even with one cycle per year. The machines should not be kept longer than their maximum life. $\bar{n}_{m_i}$ represents the maximum life of m_i machines.

Before developing general expressions for the adverse minimum challenger and defender, the following definition of symbols is made:

- d_i: Annual decrement of the available capacity of each machine at machine center i
- $STERMS^*_{m_i,k}$: The value of $STERMS(N_i)$ when N_i is an optimum cycle with m_i machines which are k years old
- $\bar{N}_{m_i}$: The maximum life of m_i machines
- B_i: The first cost of a new machine available in the market
- J_i: The current market value of a present machine at machine center i
- b_{in}: The estimated salvage ratio of a new machine with respect to B_i at the end of n years from now
- j_{in}: The estimated salvage ratio of a present machine with respect to J_i at the end of n years from now
- G_i: The gradient of operating inferiority of each machine at machine center i
- f_i: The fixed cost for carrying each machine of machine center i, which is not counted neither by operating inferiority nor $STERMS(N_i)$
- F_i: Inferiority gap between a new machine and a present one. For a new machine, F is zero. For a present machine, it is equal to the difference in operating inferiority between the new one and the present one
- x: The age of the present machines at machine center i
- r: Discount rate

Using the above symbols, the adverse minimum of m_i new machines can be expressed as follows:

(Adverse minimum of m_i new machines)

$$= \min_{n=1,2,\ldots,\bar{n}_{m_i}} [(m_i \cdot B_i)(\tfrac{a}{p})^r_n - (m_i \cdot B_i \cdot b_{in})(\tfrac{a}{f})^r_n$$

$$+ (m_i \cdot G_i)(\tfrac{a}{g})^r_n + m_i f_i + \sum_{k=1}^{n} \text{STERMS}^*_{m_i,k} (\tfrac{p}{f})^r_k (\tfrac{a}{p})^r_m] . \quad (3.17)$$

To obtain the adverse minimum of challenger, there is one more variable yet to be determined in equation 3.17. That is m_i which represents the total number of new machines to be installed at machine center i. Since the challenger is composed of m_i^* new machines which minimize the adverse minimum over all possible values of m_i, m_i^* should be found. The only way to find m_i^* seems to be to calculate the adverse minimum for each possible value of m_i and pick one value of m_i which minimizes the adverse minimum. This does not mean that an infinite number of different values of m_i should be tried. There are a lower bound and an upper bound of m_i, and the comparison of the values of the adverse minimum should be restricted within this range.

In Section C $m_i(N_i)$ was defined to be a minimum number of identical machines required at machine center i when its cycle is N_i per year. The lower bound of m_i is the number of machines required when $N_i = 1$, i.e., $m_i(N_i = 1)$. A symbol $\underline{m}_i$ is used to represent such bound and $\underline{m}_i = m_i(N_i = 1)$.

As m_i increases, less restriction is imposed on the freedom in selecting an optimum production cycle at the beginning of each year. This means that $m_i + 1$ machines can realize a smaller or at most an equal value of $STERMS(N_i)$ than m_i machines each year. In other words, $STERMS^*_{m_{i+1},k} \leq STERMS^*_{m_i,k}$ for $k = 1,2,\ldots,\bar{n}_{m_i}$. However, there is a limit value of m_i beyond which $STERMS^*_{m_{i+1},k} = STERMS^*_{m_i,k}$ for $k = 1,2,\ldots,\bar{n}_{m_i}$ and $\bar{n}_{m_{i+1}} = \bar{n}_{m_i}$. Since nothing is going to be gained by increasing the value of m_i beyond the limit, such limit is the upper bound of m_i. A symbol $\bar{m}_i$ is used to represent the upper bound. Hence, $\underline{m}_i \leq m_i^* \leq \bar{m}_i$.

Now it is possible to express the adverse minimum of challenger.

(Adverse minimum of challenger)

$$= \min_{m_i = \underline{m}_i,\ldots,\bar{m}_i} \{\min_{n=1,2,\ldots\bar{n}_{m_i}} [(m_i \cdot B_i)(\tfrac{a}{p})^r_n - (m_i \cdot B_i \cdot b_{in})(\tfrac{a}{f})^r_n$$

$$+ (m_i \cdot G_i)(\tfrac{a}{g})^r_n + m_i \cdot f_i + \sum_{k=1}^{n} STERMS^*_{m_i,k} (\tfrac{p}{f})^r_k (\tfrac{a}{p})^r_n]\}$$

$$= \min_{n=1,2,\ldots,\bar{n}_{m_i^*}} [(m_i^* \cdot B_i)(\tfrac{a}{p})^r_n - (m_i^* \cdot B_i \cdot b_{in})(\tfrac{a}{f})^r_n$$

$$+ (m_i^* \cdot G_i)(\tfrac{a}{g})^r_n + m_i^* \cdot f_i + \sum_{k=1}^{n} STERMS^*_{m_i^*,k} (\tfrac{p}{f})^r_k (\tfrac{a}{p})^r_n] \quad (3.18)$$

The adverse minimum of defender which is composed of m_i machines that are x years old can be expressed as follows:

(Adverse minimum of defender)

$$= \min_{n=1,2,\ldots,(\bar{n}_{m_i} - x)} [(m_i \cdot J_i)(\tfrac{a}{p})^r_n - (m_i \cdot J_i \cdot j_{in})(\tfrac{a}{f})^r_n$$

$$+ F_i + (m_i \cdot G_i)(\tfrac{a}{g})^r_n + m_i \cdot f_i + \sum_{k=1}^{n} \text{STERMS}^*_{m_i, x+k}(\tfrac{p}{f})^r_n \cdot (\tfrac{a}{p})^r_n]$$

(3.19)

In equations 3.17 and 3.18, $(\frac{a}{p})^r_n$, $(\frac{a}{f})^r_n$, $(\frac{a}{g})^r_n$ and $(\frac{p}{f})^r_k$ are the interest factors being used by G. W. Smith (1973).

a. The beginning production capacity of machine center i

The optimum beginning production capacity of machine center i is m^*_i. The defender in this case is the status quo which means no production at all at machine center i. Since it is mandatory to produce parts at machine center i, choosing a challenger is the only alternative.

An example is presented below to clarify the meanings of the symbols of equations 3.17 and 3.18 and the underlying logic of finding m^*_i.

Example 2) There are 7 different parts to be processed at machine center i. The following are the data given.

i) $STERMS(N_i) = 3000\ N_i + \frac{108000}{N_i}$, $N_i^* = 6$

ii) $\bar{N}_i = 8$

iii) (Time unit: year)

Parts	Processing time per year	Setup time per cycle	The consumption of the available capacity by each part at a given cycle $N_i = 1$	2	3	4	5	6
A	.1	.04	.14	.18	.22	.26	.3	.34
B	.1	.04	.14	.18	.22	.26	.3	.34
C	.1	.04	.14	.18	.22	.26	.3	.34
D	.1	.04	.14	.18	.22	.26	.3	.34
E	.2	.04	.24	.38	.32	.36	.4	.44
F	.2	.04	.24	.38	.32	.36	.4	.44
G	.2	.04	.24	.38	.32	.36	.4	.44

iv) Annual capacity decrement $d_i = .1$

v) $B_i = 20{,}000$, $b_{i1} = .4$, $b_{i2} = .2$, $b_{i3} = .1$, $b_{ik} = 0$ for $k \geq 4$

vi) $G_i = 1{,}000$, $f_i = 1{,}000$, $r = 10\%$

Solution procedures)

Step 1) Do a balanced loading beginning with $m_i(N_i = 1)$.

$m_i(N_i = 1) = 2$ machines.

The case of 2 machines

n and available capacity	1(1.0)	2(.9)	3(.8)	4(.7)
optimum N_i	$N_i = 3$	$N_i = 2$	$N_i = 1$	$N_i = 1$
Balanced loading and capacity required for each machine	.88(ABCD)	.82(ABCG)	.66(ABCG)	.66(ABCG)
	.96(EFG)	.74(EFD)	.62(EFD)	.62(EFD)

The case of 3 machines

n and available capacity	1(1.0)	2(.9)	3(.8)	4(.7)	5(.6)	6(.5)
optimum N_i	N_i=5	N_i=5	N_i=4	N_i=3	N_i=2	N_i=1
Balanced loading and capacity required for each machine	.9(ABC)	.9(ABC)	.78(ABC)	.66(ABC)	.54(ABC)	.42(ABC)
	.7(DE)	.7(DE)	.62(DE)	.54(DE)	.46(DE)	.38(DE)
	.8(FG)	.8(FG)	.72(FG)	.64(FG)	.56(FG)	.48(FG)

A similar loading scheme will be applied to each value of m_i until it reaches its upper bound which is 7 in this example.

Note that $\bar{n}_2 = 4$ and $\bar{n}_3 = 6$. $STERMS^*_{3,2}$, for example, is $STERMS(N_i{=}5) = 3000 \cdot 5 + \frac{108000}{5} = 60{,}000$. In the loading process, the production capacity required at each machine should not be bigger than the available capacity at a given year.

Step 2) By using equation 3.17, find the adverse minimum of m_i new machines for each possible value of m_i.

The case of 2 machines

n	1	2	3	4
The value of equation 3.17 for a given n	75,000	74,334	88,673	96,153

The case of 3 machines

n	1	2	3	4	5	6
The value of equation 3.17 for a given n	81,600	69,885	65,449	65,048	66,680	74,628

A similar calculation will disclose the adverse minimum of m_i new machines for each value of m_i. The summary of the results is given below.

m_i	2	3	4	5	6	7
Adverse minimum	74,334	65,048	69,925	76,697	84,671	90,172

Step 3) Find m_i^* which minimizes the adverse minimum. From the summary of the results in step 2, $m_i^* = 3$. Hence, the optimum beginning production capacity of machine center i is 3 machines and its adverse minimum is \$65,048. The dollar amount, \$65,048, is the lowest combined time-adjusted average of capital cost, operating

inferiority, and STERMS(N_i). The economic life of the 3 machines is 4 years at which the adverse minimum is realized.

b. An optimum replacement policy at the beginning of a certain year Suppose that machine center i is composed of m_i machines which are x years old at the beginning of a certain year. The decision on replacement depends on the adverse minimum of defender which is the x years old machines and that of challenger which is composed of m_i^* new machines. If the former is bigger than the latter, the replacement of the m_i old machines with m_i^* new machines is an optimum policy. Otherwise no replacement is an optimum policy. No conclusive statement can be made on the relationship between m_i and m_i^*. They can be the same or can be different depending on the data available. Whatever the values of m_i and m_i^* might be, following the optimum replacement policy at the beginning of each year will guarantee an optimum production capacity at machine center i all the time.

The adverse minimum of defender can be obtained from equation 3.19. The adverse minimum of challenger can be obtained by following the solution procedures given in Example 2.

4. Optimum replacement policy with budgeting constraints

When there is some budgeting constraints, it may not be possible to replace all the machines of machine center i at a same time. In this case, it is possible that x machines are replaced with y machines where x and y are non-negative integers. However, it is found that the replacement of x machines with y new machines requires a couple more assumptions and approximations on the setup cost and inventory holding cost which will let the analysis digress too much. While leaving the subject as a further research, the analysis is limited to a case where all the machines of machine center i are replaced at a same point of time. Under this circumstance, several machine centers whose replacements are indicated by the optimum replacement policy compete against each other for a limited budget at a beginning of each year.

The problem can be formulated in an Integer Programming with an appropriate objective function. Since the urgency of replacement is reflected by the excess of the adverse minimum of defender over that of challenger, the objective function is the summation of the excess amounts of the machine centers whose replacements are indicated. An optimum policy should identify the machine centers to be replaced and minimize the objective function.

Before setting up the linear objective function and constraints, the following definition of symbols is made.

AMC_i: The adverse minimum of challenger of machine center i

AMD_i: The adverse minimum of defender of machine center i

Ω: The set of machine centers at which $AMD_i - AMC_i$ is positive

L: Capital available for replacement at the beginning of a year

X_i: Replacement decision variable whose value is 1 or 0. It is 0 if m_i old machines of machine center i are replaced with m_i^* new machines. Otherwise it is 1

Using the above symbols the problem can be formulated as follows:

$$\text{Min} \sum_{i\in\Omega} X_i(AMD_i - AMC_i)$$

S. T.

$$\sum_{i\in\Omega} (1 - X_i) \cdot m_i^* \cdot B_i \leq L \qquad (3.20)$$

The problem can be solved by several different approaches. One of them is the MINT Algorithm which is based on the Land and Doig (1960) method. The algorithm is programmed in the FORTRAN (Kuester and Mize, 1973).

IV. STOCHASTIC CASE

A. Production Lead Time and WIP Inventory

1. Introduction

The production system to be discussed in this chapter is different from the one covered in Chapter III in several aspects. The system discussed in this section is stochastic. Even though the processing sequences of N parts are predetermined as before, the time required for production and moving among M machine centers for each part is stochastic. In addition the yearly demand of each part is stochastic with a time invariant mean. In the previous chapter each machine center processed the parts cyclically. Each machine center of the production system in this chapter processes the parts by lots and the size of the lots for each part is predetermined. The service discipline at each machine center is first-come-first-served.

Once each lot has finished its final operation, it will be stored temporarily at an area called Finished Piece Parts Storage. The demands for the finished piece parts occurring in the assembly line are satisfied from the storage. Production orders to the production floor are generated based on a lot size-reorder point model. Some of the assumptions made in Chapter III are retained in the stochastic production system of this chapter. They are assumptions 10, and 17. The relaxation of assumption 16 does not mean lot splitting. Individual units contained in one particular lot will be moved together until the lot arrives in the Finished Piece Parts Storage.

While the WIP inventory in Chapter III was referred to the parts on the production floor only, in this chapter it refers to parts on the production floor and in the Finished Piece Parts Storage.

The average level of the WIP inventory of this stochastic production system is a function of many different factors. However, it seems that the production lead time of the parts is the most important factor to the average level. Here the production lead time of a part is defined as the time spent on the production floor by the part. Since the effect of the production lead time on the average level of the WIP inventory is crucial, it is discussed in detail in the next section.

In Section 3 a brief discussion on the Theory of Queues and its applicability to the production system of this chapter is presented.

Ancker and Gafarian (1961) solved a queueing system with multiple Poission inputs and exponential service times by using recursion relations for the steady-state probabilities of n in queue and some type in service. A different approach in solving the same queueing system seems to be a little bit simpler and quicker in arriving at the same final result.

2. Production lead time and WIP inventory

The WIP inventory is composed of two different groups. One group is the WIP inventory on the production floor. The other is in the Finished Piece Parts Storage. For the sake of convenience, the former is called the WIP inventory in production and the latter is called the WIP inventory in storage. The effects of the production lead time on these two WIP inventories are different and they are discussed separately.

<u>a. Production lead time and WIP inventory in production</u> When the yearly demand of a part is distributed with a time invariant mean, then the average level of its WIP inventory in production in terms of unit-years is proportional to the average production lead time of the part. The size of lots, however, does not affect the average level of the inventory. It is easy to show the validity of the statement by the following relationship.

(Average WIP in production in unit-years)

$$= \frac{D}{Q} \cdot Q \cdot \text{(Average production lead time)} \qquad (4.1)$$

where D represents the yearly mean demand and Q represents the size of one lot. Since D is assumed to be a constant, the level of the inventory is directly proportional to the average production lead time.

The average level of the WIP inventory of a part in terms of \$-years is not necessarily proportional to its average production lead time. Since the unit dollar value of the part changes at each machine center, it is necessary to know the average time spent at each machine center as well as the unit dollar value of the part to find the average inventory level in terms of \$-years. Nevertheless, it is possible to approximate the level if the initial unit dollar value and the final unit dollar value of the part are available. By assuming that the arithmetic mean of these two unit dollar values represents the unit dollar value of the part during its production,

(Average WIP in production in \$-years)

$$= \frac{D}{Q} \cdot Q \cdot V_{mean} \cdot \text{(Average production lead time)}$$

$$= D \cdot V_{mean} \cdot \text{(Average production time)} \qquad (4.2)$$

where V_{mean} represents the arithmetic mean.

From equation 4.2 the average level of the WIP inventory in production of a part in terms of \$-years is proportional to its average production lead time.

<u>b. Production lead time and WIP inventory in storage</u> The average level of the WIP inventory in storage of a part in terms of unit-years is a complex function of its demand distribution, its production lead time distribution and the inventory policy being used at the storage. Even with a given inventory policy and a demand distribution, the functional relationship between the average level of inventory in storage and the production lead time is still a complicate one. The discussion on the relationship is limited to the stochastic production system which has been briefly described in Section A-1. The inventory policy of the Finished Piece Parts Storage of the production system is a lot size-reorder point with backorders allowed. The size of lots for each part is predetermined.

The probability of being out of stock is predetermined also for each part. The demand distribution of each part is a Poisson distribution with a known mean.

It is possible to calculate the average level of the inventory in storage in two different ways. One is by an approximate treatment and the other is by an exact formula. Both ways are well described by Hadley and Whitin (1963).

One key assumption of the approximate treatment is that there is never more than a single order outstanding. This assumption simplifies the situation a great deal. The treatment ignores the expected backorders over time in calculating the average level of on-hand inventory. Before presenting the relationship between the production lead time and the WIP inventory in storage of a part, the following definition of symbols is made.

Q: Lot size

r: Reorder point

D: Yearly mean demand rate

$p(x;\lambda)$: The demand distribution during unit time. It is a Poisson with mean λ.

$f(t)$: Marginal density function of production lead time

μ_L: Mean production lead time

σ_L: Standard deviation of production lead time

$h(x)$: Marginal density function of demand during production lead time. $h(x)$ is a discrete distribution.

μ: Mean demand during production lead time

σ: Standard deviation of demand during production lead time

$H(x)$: Complementary cumulative of $h(x)$

P_{out}: Probability of being out of stock at any given point of time

The average level of the WIP inventory in storage of the part in terms of unit-years can be calculated as follows using the above symbols:

(Average WIP in storage in unit-years)

$$= \frac{Q}{2} + r - \mu \tag{4.3}$$

where

$$P_{out} = \frac{1}{Q} \cdot \sum_{x=r}^{\infty} (x - r)\, dx$$

$$= \frac{1}{Q} \left[\sum_{x=r}^{\infty} x \cdot h(x) - rH(r) \right] \tag{4.4}$$

If the distribution of demand and that of production lead time are independent of each other, then

$$\mu = \mu_L \cdot \lambda \tag{4.5}$$

$$\sigma^2 = \mu_L \cdot \lambda + \sigma_L^2 \cdot \lambda^2 \tag{4.6}$$

$$h(x) = \int_0^{\infty} p(x;\lambda t) \cdot f(t)\, dt \tag{4.7}$$

If $h(x)$ is assumed to be a normal distribution with mean μ and standard deviation σ, then equation 4.4 can be written as

$$P_{out} = \frac{1}{Q} [\sigma \phi (\frac{r-\mu}{\sigma}) - (r-\mu) \Phi (\frac{r-\mu}{\mu})] \qquad (4.8)$$

where $\phi(w)$ and $\Phi(w)$ are the density function and the complementary cumulative distribution function of the standardized normal distribution respectively.

In the exact formula the assumption that there is never more than a single order outstanding is relaxed. However the two key assumptions of the exact formula with variable production lead time are that the production lead time of each lot is independent of the others and that orders do not cross. These two assumptions are contradicting each other. The reason why these two assumptions are made simultaneously in the exact formula in spite of their contradiction is that it is very difficult to deal with a model having only one of the two assumptions. However, in the real world the interval between the placing of orders is usually large enough that there is essentially no interaction between orders, to a good approximation, the two assumptions can be made simultaneously (Hadley and Whitin, 1963).

The average level of the WIP inventory in storage of the part in terms of unit-years by exact formula is

(Average WIP in storage in unit-years)

$$= \frac{Q+1}{2} + r - \mu + B(Q, r) \qquad (4.9)$$

where

$$P_{out} = \frac{1}{Q} \left[\sum_{y=0}^{\infty} H(r+1+y) - \sum_{y=0}^{\infty} H(r+Q+y+1) \right] \tag{4.10}$$

and

$$B(Q,r) = \frac{1}{Q} \left[\sum_{y=0}^{\infty} y \cdot H(r+1+y) - \sum_{y=0}^{\infty} y \cdot H(r+Q+y+1) \right] \tag{4.11}$$

Hadley and Whitin derived equation 4.9 for the case of a constant lead time. In this case $h(x)$ is a Poisson distribution. However it is also valid for the case of a variable lead time. In this case $h(x)$ is not necessarily a Poisson distribution. The proof of this is given in Appendix B.

If $h(x)$ is assumed to be a normal distribution with mean μ and standard deviation σ, then equations 4.10 and 4.11 can be rewritten as

$$P_{out} = \frac{1}{Q} \left[\sigma\, \phi\left(\frac{r-\mu}{\sigma}\right) - (r-\mu)\, \Phi\left(\frac{r-\mu}{\sigma}\right) \right]$$

$$- \frac{1}{Q} \left[\sigma\, \phi\left(\frac{r+Q-\mu}{\sigma}\right) - (r+Q-\mu)\, \Phi\left(\frac{r+Q-\mu}{\sigma}\right)\right] \tag{4.12}$$

and

$$B(Q,r) = \frac{1}{Q} \left\{ \frac{1}{2}[\sigma^2 + (r-\mu)^2]\, \Phi\left(\frac{r-\mu}{\sigma}\right) - \frac{\sigma}{2}(r-\mu)\, \phi\left(\frac{r-\mu}{\sigma}\right)\right\}$$

$$- \frac{1}{Q} \left\{\frac{1}{2}[\sigma^2 + (r+Q-\mu)^2\, \Phi\left(\frac{r+Q-\mu}{\sigma}\right) - \frac{\sigma}{2}(r+Q-\mu)\, \phi\left(\frac{r+Q-\mu}{\sigma}\right)\right\} \tag{4.13}$$

The average level of the WIP inventory in storage of the part in terms of \$-years by either the approximate treatment or the exact formula is

(Average WIP in storage in \$-years)

$$= \text{(Average WIP in storage in unit-years)} \cdot V_{final} \tag{4.14}$$

where V_{final} represents the final unit dollar value of the part.

The difference between the approximate treatment and the exact formula in calculating the average level of the WIP inventory in storage is discussed in a later section.

3. Queueing theory and its applicability to the stochastic production system

Since the stochastic production system of this chapter belongs to a particular class of queueing system, namely Queueing Networks, it is worthwhile to examine the currently existing Queueing Theory and its applicability to the problem. Especially one entity of main interest is the total waiting time spent by a customer in a system of queueing networks. This is so because it has been shown that the average level of the WIP inventory of a part in production as well as in storage is closely related with its production lead time which corresponds to the total waiting time.

In his article, "Queueing Theory: The State-of-the-Art", Rosenshine (1975) has done an extensive survey on the theory of queue and its applications. Especially he looked at the recent literature of the subject very closely and summarized the current body of the theory and its applications in a concise form. One of his conclusions based on the survey is that the current knowledge on queueing networks is very limited and the area is wide open. He found that solving even simple networks by a rigorous analysis presented a lot of difficulties. He considered approximation solutions the most promising general approach to solving queueing networks and pointed out an apparent shift in the theory from the rigorous analytical approach to approximation analyses and analytic-simulation hybrid methods.

In his article, "Random Flow in Queueing Networks: A Review and Critique", Disney (1975) exclusively reviewed various aspects and problems in queueing networks and has done an extensive survey on the subject. One of his introductory remarks is "The point is that, compared to the state-of-the-art in single-server queueing, knowledge about multiple server queues is in a rather primitive state." He found three basic approaches in solving queueing networks. The first one is to study the vector valued process of queueing networks. The major impediment in this case is dimensionality. The approach requires the tools of linear algebra and matrix theory for solving large scale systems of equations. The second one is to decompose queueing networks into subnetworks so as to use the wealth of known results for the single-server queues. One major problem in this case is how to decompose and recombine the subnetworks in such a way that the integrality of the networks and the original stochastic properties of the flows in the networks are not destroyed. The other problem, which is more basic, is to solve scalar valued non-Markov processes whose random variables may depend on many other random variables in the networks. The third one is to study queueing networks by computer simulation. One question in this case is how often to sample the simulated output so as to get nearly independent estimates. In the summary of the paper, Disney mentioned two new areas which seem to be recent developments. They are 1) finding approximate numerical solutions to the steady state equations and 2) network reduction techniques through the study of the flow graphs that are approximately the same as the flow graphs of the original network.

The author has searched through the queueing literature to find an analytical model which could be applied to the study of the stochastic production system of this chapter. Unfortunately, the search was not successful. The only alternative seemed to be computer simulation and this is what actually has been done. A more detailed discussion on the computer simulation is presented in a later section.

Before closing this section, two articles by Jackson (1957 and 1963) deserve attention. Both articles were written with an intention to apply queueing theory to job shop type of production system.

In his first article (1957), the machine shop studied had the following features: (1) Each department is a multiserver system where servers are arranged parallel and waiting jobs are pooled in a single line, (2) the service time distribution of each server of the department is exponential with a same mean, (3) arrivals at a given department come both from other departments in the shop and from outside the shop, (4) the arrival distribution from outside to any department is Poisson with a given mean, and (5) the probability that a finished job from a given department goes to some specified department or out of the system is given.

Jackson proved a theorem which says that as long as the system is steady state, it behaves as if its departments were independent multiserver queueing systems. The average queue length at each department could be acquired very easily. The approach taken in arriving at the result was the first one mentioned by Disney.

In his second article (1963), the jobshop-like queueing system investigated had the following features: (1) The service time distribution of each department is exponential with a mean which varies almost arbitrarily with the queue length there, (2) the arrival distribution from outside to the system is Poisson with a mean which varies almost arbitrarily with the total number of customers already in the system, (3) each arrival is assigned a routing which is generated by a specified routing generation process, and (4) service discipline at each machine center is random.

Jackson proved three theorems in this paper. The first one says that if the value of a particularly specified function is strictly positive, then a unique equilibrium state probability distribution exists for the system. The second one is similar to the first one for the case where the immediate injection of a new customer is triggered whenever the total number of customers falls below a specified limit, or where a service is deleted if a queue length grows beyond a specified maximum length. The second theorem was extended to cover a case where the total number of customers was held fixed. The third one is also similar to the first one for the case where the rate of customer arrivals is constant. He showed the actual application of his theorems for solving couple simple networks in examples. Compared to the previous paper, it seems to be more difficult to obtain average queue length at each department. This is much more so as the number of departments increases. The approach taken in arriving at the results was similar to the one in his previous paper.

Although the physical characteristics of the two systems which Jackson studied are similar to those of the stochastic production system, the main difference stems from the probability distributions of arrivals and services. To be specific, the arrival distribution and the service distribution of the stochastic production system are Gamma and Normal respectively. This is the reason why the results of Jackson could not be applied to the problem. However, those two papers dealt with queueing networks which are very similar to the network of the production system of this chapter.

4. Alternative approach in solving one-server queueing system with multiple Poisson inputs and exponential service times

Ancker and Gafarian (1961) solved a single server queueing system for N different types of customers having independent Poisson arrivals with rates λ_i, $i = 1, \ldots, N$ and exponential service times with rates μ_i, $i = 1, \ldots, N$. The service discipline was first-come-first-served. There was no limit for the length of queue. A recursion relation was derived for the steady-state probability of n in queue. They also derived a recursion relation for the steady-state probability that some member of a particular class is in service and n of any class are in queue. From these relationships and two moment generating functions, the expected number of customers in the system is calculated as follows:

$$L = \sum_{i=1}^{N} \frac{\lambda_i}{\mu_i} + \frac{\lambda \sum_{i=1}^{N} \frac{\lambda_i}{\mu_i}}{1 - \sum_{i=1}^{N} \frac{\lambda_i}{\mu_i}}$$

where L stands for the expected number of customers in the system.

The same result has been achieved using a different approach. The main idea of this approach is to calculate the ratios among moment generating functions and use these ratios in finding the expected number of customers in the system. This approach seems to be a little easier to understand and quicker to arrive at the final result. This approach was applied to a multiple server queueing system but unfortunately it failed. Since the incoming jobs to a production system are usually composed of different classes in terms of their input rates and service rates, the queueing system with heterogeneous inputs is an important type of queueing system for the analysis of production systems, particularly for a jobshop type production system. Consequently, multiple server case deserves attention and it should be studied further.

Before getting into the details of this approach, the following symbols and functions are defined.

N: The total number of different classes of jobs in terms of their input rates and service rates

P_n: Steady state probability that n jobs of any class are in the system where n includes the job being served if there is any

P_{in}: Steady state probability that a job of i class is being served and $n-1$ jobs of any class are waiting

λ_i: Input rate of the jobs of i class

μ_i: Service rate of the jobs of i class

λ: $\lambda = \sum_{i=1}^{N} \lambda_i$

δ_i: $\delta_i = \frac{\lambda_i}{\lambda}$

z: A real number in the range $0 < z \leq 1$

$F_i(z)$: A moment generating function defined as

$$F_i(z) = \sum_{n=1}^{\infty} z^n P_{in}$$

F(z): A moment generating function defined as

$$F(z) = \sum_{n=1}^{\infty} z^n P_n$$

By definition,

$$P_n = \sum_{i=1}^{N} P_{in} \qquad (n=1,2,\ldots) ,$$

$$\sum_{i=1}^{N} \delta_i = 1, \text{ and}$$

$$F(z) = \sum_{n=1}^{\infty} z^n \sum_{i=1}^{N} P_{in} = \sum_{i=1}^{N} F_i(z).$$

The set of steady-state equations established by the conservation of flow is

$$\sum_{i=1}^{N} \mu_i P_{i1} = \lambda P_0, \tag{1}$$

$$\lambda_i P_0 + \delta_i \sum_{j=1}^{N} \mu_j P_{j2} = \lambda P_{i1} + \mu_i P_{i1}, \quad (i=1,\ldots,N)$$

(2)

$$\lambda P_{i,n-1} + \delta_i \sum_{j=1}^{N} \mu_j P_{j,n+1} = \lambda P_{in} + \mu_i P_{in},$$

$$(i=1,\ldots,N,\ n=2,3,\ldots) \tag{3}$$

$$\sum_{n=0}^{\infty} P_n = 1\ . \tag{4}$$

By summing both sides of equation 2 over $i=1,\ldots,N$,

$$\lambda P_1 = \sum_{j=1}^{N} \mu_j P_{j2}\ .$$

Similarly, by summing both sides of equation 3 over $i=1,\ldots,N$ for $n=2,3,\ldots$, respectively, it can be shown that

$$\lambda P_n = \sum_{j=1}^{N} \mu_j P_{j,n+1}\ . \quad (n=0,1,2,\ldots) \tag{5}$$

By summing both sides of equation 5 over $n=0,1,2,\ldots$,

$$\lambda P_0 + \lambda F(1) = \sum_{j=1}^{N} \mu_j F_j(1)\ . \tag{6}$$

By multiplying both sides of equation 5 with z^{n+1} for $n=0,1,2,\ldots$, respectively,

$$\lambda z^{n+1} P_n = \sum_{j=1}^{N} \mu_j z^{n+1} P_{j,n+1}. \quad (n=0,1,2,\ldots) \tag{5.a}$$

By summing both sides of equation 5.a over $n=0,1,2,\ldots$,

$$\lambda z P_0 + \lambda z F(z) = \sum_{j=1}^{N} \mu_j F_j(z). \tag{7}$$

Division of equation 6 and 7 with $F(1)$ and $F(z)$ respectively gives

$$\frac{\lambda P_0}{F(1)} + \lambda = \sum_{j=1}^{N} \mu_j \frac{F_j(1)}{F(1)} , \tag{6.a}$$

$$\frac{\lambda z P_0}{F(z)} + \lambda z = \sum_{j=1}^{N} \mu_j \frac{F_j(z)}{F(z)} . \tag{7.a}$$

If the ratio, $\frac{F_j(1)}{F(1)}$, in equation 6.a can be expressed in terms of μ_j, δ_j and λ, then P_0 can be solved. Likewise if the ratio, $\frac{F_j(z)}{F(z)}$, in equation 7.a can be expressed in terms of μ_j, δ_j, λ and z, then $F(z)$ can be solved.

By multiplying both sides of equation 2 with z,

$$\lambda_i z P_0 + \delta_i \sum_{j=1}^{N} \mu_j z P_{j2} = \lambda z P_{i1} + \mu_i z P_{i1}. \quad (i=1,\ldots,N) \tag{2.a}$$

Similarly, multiplying both sides of equation 3 with z^n for $n=2,3,\ldots$, respectively, gives

$$\lambda z^n P_{i,n-1} + \delta_i \sum_{j=1}^{N} \mu_i z^n P_{j,n+1} = \lambda z^n P_{in} + \mu_i z^n P_n.$$

$$(i=1,\ldots,N, \quad n=2,3,\ldots) \tag{3.a}$$

After summing both sides of equation 3.a over $n=2,3,\ldots$, and then adding equation 2.a, it can be shown that

$$\lambda_i z P_0 + \lambda z F_i(z) + \delta_i \sum_{j=1}^{N} \frac{\mu_j}{z} \left[F_j(z) - zP_{j1}\right]$$

$$= \lambda F_i(z) + \mu_i F_i(z). \qquad (i=1,\ldots,N) \tag{3.b}$$

By rearranging the terms, equation 3.b can be rewritten as

$$\mu_i F_i(z) \left[\left(\frac{\delta_i}{z} - 1\right) - \frac{\lambda}{\mu_i}(1 - z)\right] + \delta_i \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{\mu_j}{z} F_j(z)$$

$$= (1 - z)\, \lambda P_0 \delta_i. \qquad (i=1,\ldots,N) \tag{3.c}$$

Using matrix notation, equation 3.c is rewritten as follows:

$$
\begin{bmatrix}
(\frac{\delta_1}{z}-1)-\frac{\lambda}{\mu_1}(1-z) & \frac{\delta_1}{z} & \cdots & \frac{\delta_1}{z} \\
\frac{\delta_2}{z} & (\frac{\delta_2}{z}-1)-\frac{\lambda}{\mu_2}(1-z) & \cdots & \frac{\delta_2}{z} \\
 & & \vdots & \\
\frac{\delta_N}{z} & \frac{\delta_N}{z} & \cdots & (\frac{\delta_N}{z}-1)-\frac{\lambda}{\mu_N}(1-z)
\end{bmatrix}
\begin{bmatrix}
\mu_1 F_1(z) \\ \mu_2 F_2(z) \\ \vdots \\ \mu_N F_N(z)
\end{bmatrix}
$$

$$
= (1-z)\lambda P_0 \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_N \end{bmatrix} \tag{8}
$$

By setting z=1 in equation 8,

$$
F_i(1) = \frac{\delta_i}{\mu_i} \sum_{j=1}^{N} \mu_j F_j(1). \qquad (i=1,\ldots,N)
$$

Since $F(1) = \sum_{i=1}^{N} F_i(1)$,

$$
\frac{F_i(1)}{F(1)} = \frac{\delta_i}{\mu_i \sum_{i=1}^{N} \frac{\delta_i}{\mu_i}} \qquad (i=1,\ldots,N) \tag{8.a}
$$

Equation 8.a is the desired ratio to be substituted in equation 6.a.

After substitution and rearranging terms,

$$P_0 = 1 - \sum_{i=1}^{N} \frac{\lambda_i}{\mu_i} , \tag{10}$$

where $\sum_{i=1}^{N} \frac{\lambda_i}{\mu_i} < 1.$

The ratio, $\frac{F_i(z)}{F_j(z)}$, for $i=1,\ldots,N$ and $j = 1,\ldots,N$ can be obtained from equation 8 and that is

$$\frac{F_i(z)}{F_j(z)} = \frac{\delta_i\ [\mu_j + \lambda(1-z)]}{\delta_j\ [\mu_i + \lambda(1-z)]} . \tag{8.b}$$

Since $F(z) = \sum_{i=1}^{N} F_i(z)$,

$$\frac{F(z)}{F_j(z)} = 1 + \sum_{\substack{i=1\\ i\neq j}}^{N} \frac{F_i(z)}{F_j(z)} = 1 + \sum_{\substack{i=1\\ i\neq j}}^{N} \frac{\delta_i\ [\mu_j + \lambda(1-z)]}{\delta_j\ [\mu_i + \lambda(1-z)]}$$

$$= \frac{\sum_{k=1}^{N} \delta_k \prod_{\substack{i=1\\ i\neq k}}^{N} [\mu_i + \lambda(1-z)]}{\delta_j \sum_{\substack{i=1\\ i\neq j}}^{N} [\mu_i + \lambda(1-z)]} . \quad (j=1,\ldots,N) \tag{8.c}$$

Equation 8.c is the desired ratio to be substituted in equation 7.a. By substituting the ratio and then multiplying the right hand side of equation 7.a with $\frac{1}{\sum_{i=1}^{N} [\mu_i + \lambda(1-z)]}$,

$$F(z) = \frac{P_0}{\frac{K}{\lambda z} - 1} , \tag{11}$$

$$\text{where } K = \frac{\sum_{j=1}^{N} \frac{\mu_j \delta_j}{[\mu_j + \lambda(1-z)]}}{\sum_{k=1}^{N} \frac{\delta_k}{[\mu_k + \lambda(1-z)]}} .$$

Now the expected number of jobs in the system, L, is

$$L = F'(z)\Big|_{z=1} = \frac{-P_0 \left(\frac{K}{\lambda z}\right)'}{\left(\frac{K}{\lambda z} - 1\right)^2} .$$

By defining
$$R = \sum_{i=1}^{N} \frac{\lambda i}{\mu_i} ,$$

$$\left(\frac{K}{\lambda z}\right)'_{z=1} = \frac{R^2 - [\sum_{i=1}^{N} \frac{\lambda_i(\mu_i + \lambda)}{\mu_i^2}]}{R^2} = \frac{R^2 - (R + \lambda R)}{R^2} ,$$

$$\left(\frac{K}{\lambda z} - 1\right)^2_{z=1} = \left(\frac{1}{R} - 1\right)^2 = \frac{(1 - R)^2}{R^2} .$$

Since $P_0 = 1 - R$ from equation 10,

$$L = F'(z)_{z=1} = \frac{(1 - R)\frac{(R + \lambda R - R^2)}{R^2}}{\frac{(1 - R)^2}{R^2}} = \frac{R(1 - R) + \lambda R}{1 - R}$$

$$= R + \frac{\lambda R}{1 - R} = \sum_{i=1}^{N} \frac{\lambda_i}{\mu_i} + \frac{\lambda \sum_{i=1}^{N} \frac{\lambda_i}{\mu_i}}{1 - \sum_{i=1}^{N} \frac{\lambda_i}{\mu_i}} \tag{12}$$

Before closing this section, one paper by Kotiah and Slater (1973) is to be mentioned. They solved a queueing system of two servers for two types of customers with different arrival rates (λ_i, i=1,2) and service rates (μ_i, i=1,2). Their arrival distribution and service distribution are independent Poisson and exponential, respectively. The service discipline is first-come-first-served. Two moment generating functions are defined as follows:

$$\Psi_{ij}(z) = \sum_{n=2}^{\infty} P_{ij:n}\, z^{n-2}, \quad (i=1,2, \quad j=1,2)$$

$$\Psi(z) = \sum_{n=2}^{\infty} P(n) z^{n-2} ,$$

where $P_{ij:n}$ is the steady state probability that there are n customers in the system including a type i customer at server 1 and type j customer at server 2 and $P(n)$ is the steady state probability that there are n customers in the system. By defining $Y_{ij} = z(\lambda z - \lambda - \mu_i - \mu_j) + \delta_i\mu_i + \delta_j\mu_j$, $R = \mu_1\delta_2 P_{10:1} - \mu_2\delta_1 P_{20:1}$ and $x = 1 - z$, $\Psi(z)$ is expressed in terms of λ, λ_i, δ_i, μ_i, Y_{12}, R, x, P(1), $P_{10:1}$, and $P_{20:1}$. By using the condition $0 < \Psi(z) < 1$, the ratio of $\frac{P_{10:1}}{P_{20:1}}$ is obtained at $z = z_0$ and the expected number of customers in the system is obtained. Since the details of their development were not given in the paper, the author could not quite follow their method, and it has not been cleared to him yet.

B. Optimum Trade-Off among WIP Inventory, Production Orders and Service Rates

1. Introduction

The main objective of this section is to find empirical functional relations between the mean and variance of the production lead time of each part and the number of production orders to the shop and the service rates of machine centers in two hypothesized production systems. The two systems are designed in such a way that at every machine center the ratio of the average required machine hours and the total available machine hours per year is the same. Because of this the work assignments among different machine centers are well balanced.

The systems are modeled in GPSS language and simulated. During the simulation, the mean and variance of the production lead time of each part are observed at various combinations of the number of production orders and service rates. In one system the number of production orders is decreased gradually by setting aside a portion of the total incoming orders to the shop. No change is made on service rates. In the other system not only the number of orders is decreased, but also the service rates of all machine centers are increased by a fixed rate. The functional relations are obtained by regression analysis based on the data collected from the simulation.

To find an optimum trade-off point among WIP inventory, the number of production orders and service centers, a fixed penalty cost is assigned to setting aside one percent of the total incoming production orders per year and increasing one percent of the initial service rates at all machine centers respectively.

2. Two hypothesized production systems

The first system which is called system 1 produces 11 different parts with 5 machine centers. The number of identical machines at each machine center is 2. There is one waiting line in front of each machine center and incoming parts are served based on first-come-first-served discipline. There is no limit to queue length. The production sequence as well as the lot size of each part is pre-determined. The processing time of each part at each machine center follows normal distribution. The moving time of each part from one center to a subsequent center is also normally distributed. The demand of each part occurs at the Finished Piece Parts Storage and it follows Poisson distribution. The inventory policy at the storage is a lot size-reorder point model. The probability that a part is out of stock at any given time is 0.95. The total available machine hours per year at each machine is 11,520 where the time unit is 10 minutes. This figure is obtained by assuming that there are 20 working days each month and 8 working hours each day.

The second system, which is called system 2, produces 8 different parts with 4 machine centers. The number of identical machines at each machine center is 1, 1, 2 and 3 for machine centers 1, 2, 3 and 4, respectively. The other features of the system are the same as system 1 except one; in system 1 the incoming parts at a given machine center are heterogeneous regarding processing time while those in system 2 are homogeneous.

Tables 4.1 and 4.2 show the system parameters as well as production sequences in both systems. In the tables Q_i and λ_i represent the lot size and the mean demand rate per unit time for part i respectively. $\Gamma_i(a,b)$ represents the interarrival time distribution of the production orders to the shop, which is Gamma with mean a and standard deviation b. In Table 4.1 the production sequence and processing time of each part are described as j(c,d) where the processing time of part i at machine center j in the sequence is normally distributed with mean c and standard deviation d. In Table 4.2 the parameters of processing time are eliminated since they are the same at a given machine center. The mean and standard deviation of processing time at each center are as follows:

Machine center 1: N(20,2)

Machine center 2: N(30,3)

Machine center 3: N(40,4)

Machine center 4: N(60,6)

Table 4.1. The system parameters and the production sequence of system 1 (Time unit: 10 minutes)

Data \ Part	Lot size, demand, production order, production sequence and processing time
1	$Q_1 = 3456$ $\lambda_1 = 15$ r_1 (230, 3.9) 1(35,3.5), 3(40,4.0), 2(30,3.0)
2	$_2 = 576$ $\lambda_2 = 2$ r_2 (288, 12.0) 1(40,4.0), 2(60,6.0), 3(50,5.0), 5(40,4.0)
3	$Q_3 = 2880$ $\lambda_3 = 10$ r_3(288, 5.4) 1(30,3.0), 2(45,4.5), 5(40,4.0), 3(50,5.0), 4(30,3.0)
4	$Q_4 = 7600$ $\lambda_4 = 20$ r_4(384, 4.4) 1(33,3.3), 5(50,5.0), 4(41,4.1), 3(33,3.3), 5(40,4.0) 2(34,3.4)
5	$Q_5 = 1536$ $\lambda_5 = 8$ r_5(192, 4.9) 1(50,5.0), 3(50,5.0), 4(45,4.5), 5(50,5.0), 4(40,4.0) 2(70,7.0), 5(40,4.0)
6	$Q_6 = 384$ $\lambda_6 = 1$ r_6(384, 19.6) 1(120,12.0), 4(144,14.4), 5(120,12.0)
7	$Q_7 = 192$ $\lambda_7 = 1$ r_7(192, 13.9) 1(20, 2.0), 5(12,1.2), 2(20,2.0), 3(18,1.8), 5(12,1.2) 4(20,2.0), 3(22,2.2)
8	$Q_8 = 864$ $\lambda_8 = 3$ r_8(288, 9.8) 1(65,6.5), 3(80,8.0), 4(30,3.0), 2(84, 8.4), 4(37,3.7)

Table 4.1 (continued)

9	$Q_9 = 1152$ $\lambda_9 = 5$ $\Gamma_9(230, 6.8)$ 1(40,4.0), 4(20,2.0), 3(40,4.0), 5(22,2.2), 4(20,2.0) 2(30,3.0)
10	$Q_{10} = 1536$ $\lambda_{10} = 4$ $\Gamma_{10}(384, 9.8)$ 1(30,3.0), 4(20,2.0), 5(30,3.0), 2(45,4.5), 3(25,2.5)
11	$Q_{11} = 329$ $\lambda_{11} = 2$ $\Gamma_{11}(165, 9.1)$ 1(60,6.0), 2(67,6.7), 3(67,6.7), 4(67,6.7), 5(67.6.7)

Moving time of each part: N(24, 2.4)

Table 4.2. The system parameters and the production sequence of system 2 (Time unit: 10 minutes)

Data / Part	Lot size, demand, production order, production sequence and processing time		
1	$Q_1 = 1440$	$\lambda_1 = 10$	$\Gamma_1(144, 3.8)$
	2, 3, 1		
2	$Q_2 = 1152$	$\lambda_2 = 7$	$\Gamma_2(165, 4.9)$
	1, 3, 4		
3	$Q_3 = 628$	$\lambda_3 = 3$	$\Gamma_3(209, 8.4)$
	1, 3, 1, 4		
4	$Q_4 = 4608$	$\lambda_4 = 16$	$\Gamma_4(288, 4.2)$
	4, 3, 1, 3, 4		
5	$Q_5 = 5120$	$\lambda_5 = 20$	$\Gamma_5(256, 3.6)$
	4, 1, 3, 4, 3, 1, 4		
6	$Q_6 = 230$	$\lambda_6 = 1$	$\Gamma_6(230, 15.2)$
	3, 4, 1, 2, 3, 4, 2		
7	$Q_7 = 1646$	$\lambda_7 = 5$	$\Gamma_7(329, 8.1)$
	1, 2, 3, 4, 1, 2, 4, 2		
8	$Q_8 = 5236$	$\lambda_8 = 15$	$\Gamma_8(349, 4.8)$
	4, 3, 2, 1, 4, 3, 2, 1, 2		
	Moving time of each part: N(20, 2.0)		

3. GPSS simulation and results

The GPSS model used for simulating the two hypothesized systems is fairly simple. The whole production sequence of a part is modeled in one independent segment. The arrival of job orders for the part is simulated by a GENERATE block in its segment. A transaction generated by the block, which is equivalent to a single lot, goes to a QUEUE block which simulates the waiting line in front of the first machine center in the production sequence of the part. When its turn comes, it moves into an ENTER block which simulates occupying one machine of the center by the lot. Without any delay the transaction goes to a DEPART block and then it goes to an ADVANCE block where it spends a time period which is equivalent to the processing time of the lot. The movement into the ENTER block by the transaction increases the number of occupied machines in the center by one. On the other hand the movement into the DEPART block by the transaction decreases the number of waiting jobs in the queue by one. When the transaction comes out of the ADVANCE block it goes to the LEAVE block to decrease the number of occupied machines by one. Then it moves to another ADVANCE block and spends a time period for moving the lot to a subsequent machine center. The transaction keeps moving through a sequence of blocks which is similar to the one just described until it arrives at the Finished Piece Parts Storage. Before it is exterminated by moving into a TERMINATE block, its total residence time in the segment, which is equivalent to the production lead time of the lot, is saved by a

MSAVEVALUE block. The sample mean and variance are printed out by a TABULATE block.

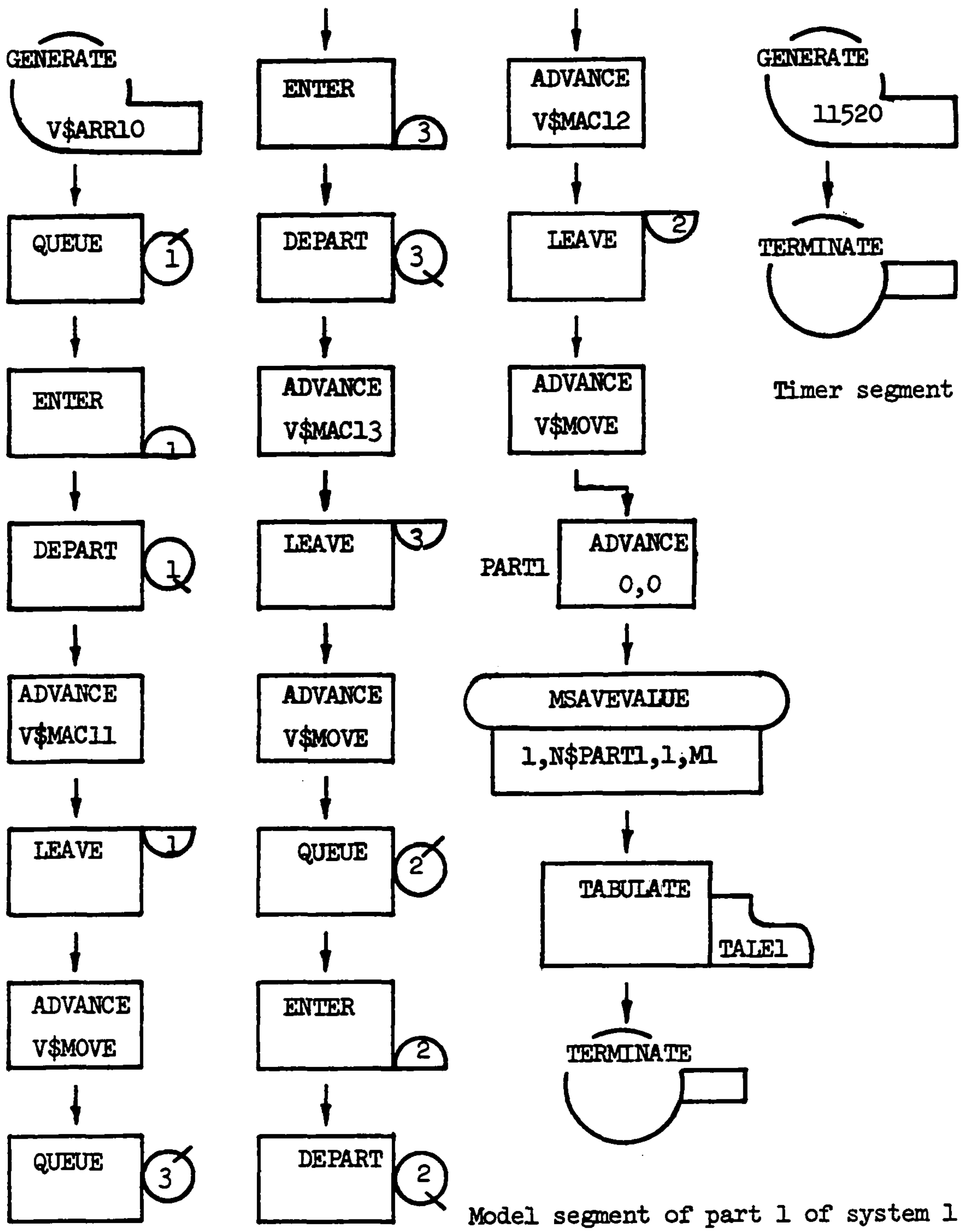
There is another model segment other than those for individual parts. This segment is a timer segment which controls the total time period of the simulation.

In each run a RESET card is used to eliminate the effects of transient period on stationary period. Also a RMULT card is used to provide a different seed value to a random number generator.

The transient period is decided by observing the variation of production lead times of individual parts from a couple of test runs. By considering the cost of computer runs and the results from the test runs, the transient period is determined to be one year. This is equivalent to 11520 time units in the simulation model where the time unit is 10 minutes. After eliminating the statistics gathered during the transient period, the model is run one more year for system 1 and two more years for system 2 to obtain required data.

Figure 4.1 shows the model segment of part 1 and the timer segment of system 1. The model segments for other parts are similar to Figure 4.1.

In Figure 4.1 V$ARR10 in the GENERATE block is the interarrival time of production orders for part 1. ARR10 is the name of a variable which is defined as 4*FN1 + 230. FN1 is a continuous GPSS function which approximates the cumulative of standard normal distribution. Although the interarrival time is Gamma distribution, it is approximated by normal distribution in the model. Hence FN1 is multiplied by the standard deviation and added by the mean of the



Timer segment

Model segment of part 1 of system 1

Figure 4.1. The block diagram for the model segment of part 1 of system 1 and the timer segment

interarrival time. Since the number to be multiplied and added to a function is limited to an integer value in GPSS, the standard deviation and the mean are rounded off to integer values.

V$MAC11, V$MAC13 and V$MAC12 and V$MOVE in ADVANCE blocks are the processing time and moving time of part 1 respectively. Their values are obtained from four different variables which are defined similarly to ARR10.

Before presenting the results of the simulation, two multipliers X_1 and X_2, which are used as two independent variables in the simulation as well as in the regression analysis, are introduced. X_1 represents the percentage to be actually processed in the shop out of the total incoming production orders per year. Hence $1.0 - X_1$ is the percentage to be set aside. X_2 represents the ratio of the actual service rates to the initial service rates at all machine centers.

In system 1 the production lead time of each part is observed as $1/X_1$ changes from 1.0 to 1.5 with interval 0.05 while X_2 is set equal to 1.0. In system 2 observations are made at 17 different combinations of X_1 and X_2. The combinations are selected in such a way that any interaction of X_1 and X_2 can be identified easily in the later regression analysis. Figure 4.2 shows the actual combinations of X_1 and X_2.

Tables 4.3 and 4.4 show the simulation results for system 1 and system 2 respectively. In both tables three numbers are presented for a given part and a given combination of X_1 and X_2. The first and the second are the sample mean and the sample standard deviation of the production lead time respectively. The last number is sample size.

X_2 / $1/X_1$	1.0	1.1	1.2	1.3	1.4	1.5
1.0	x	x	x	x	x	x
1.1	x	x	x	x		
1.2	x	x	x			
1.3	x	x				
1.4	x					
1.5	x					

Figure 4.2. The combinations of X_1 and X_2 for system 2

Table 4.3. The simulation results for system 1 (X_2 = 1.0)

Parts \ $1/X_1$	1.00	1.05	1.10	1.15	1.20	1.25
1	361.43	294.75	256.17	257.77	231.20	226.20
	55.27	32.78	40.98	45.68	36.01	37.67
	101	48	46	43	41	40
2	518.54	434.72	385.89	363.18	353.18	344.38
	53.68	46.89	54.26	37.28	41.66	45.59
	80	39	37	34	33	32
3	622.55	501.79	439.19	453.89	406.70	411.13
	69.06	60.86	59.96	51.27	62.20	53.46
	80	39	36	35	33	31
4	662.80	559.24	534.70	577.96	469.12	493.83
	55.57	82.52	64.22	59.22	51.01	58.08
	60	29	27	26	26	23
5	885.19	761.91	704.51	673.21	653.63	631.54
	70.35	76.29	66.49	69.20	61.57	53.85
	119	58	55	52	51	48
6	571.47	505.14	496.22	481.92	490.44	486.65
	44.95	38.22	36.94	33.39	40.33	32.48
	60	28	27	26	25	24
7	715.24	580.54	544.60	501.85	449.53	424.00
	82.67	77.99	87.35	78.47	69.08	50.58
	121	57	55	52	51	49
8	631.24	568.97	532.84	504.32	473.39	492.28
	56.90	55.95	43.55	52.55	41.81	48.76
	80	38	37	34	33	32
9	661.09	576.74	509.93	489.95	451.51	455.20
	61.52	67.16	79.29	47.68	45.71	68.47
	99	47	46	43	43	40
10	548.87	466.31	383.64	464.59	358.36	360.08
	45.21	61.13	49.91	79.44	45.60	59.66
	60	29	28	27	25	24
11	696.22	603.75	562.68	544.56	517.76	517.68
	51.48	48.98	51.97	45.16	41.00	46.04
	140	67	63	62	58	57

Table 4.3. (continued)

Parts \ $1/X_1$	1.30	1.35	1.40	1.45	1.50
1	230.26 43.14 39	207.43 27.82 37	233.19 36.11 36	214.32 35.97 34	198.64 34.95 33
2	329.06 33.09 31	348.34 46.50 30	343.11 33.52 28	329.67 35.91 27	329.26 37.57 27
3	377.35 40.10 31	386.73 58.32 30	385.31 41.16 29	389.39 44.58 28	366.85 41.18 27
4	461.83 46.63 23	471.64 52.58 22	455.00 38.52 21	434.70 42.32 20	416.15 30.93 20
5	618.63 51.97 46	581.42 56.33 45	595.42 58.20 43	576.66 62.26 41	565.20 44.74 40
6	482.00 40.69 23	480.95 35.86 22	457.71 21.00 21	472.71 34.45 21	464.50 26.11 20
7	415.83 50.63 46	402.53 62.46 45	392.00 52.69 43	385.24 49.67 42	380.00 48.16 41
8	465.23 37.74 31	468.17 41.63 30	458.75 46.38 28	461.79 42.81 28	448.37 37.53 27
9	423.18 56.16 39	394.24 64.55 38	398.56 53.31 36	387.03 50.84 35	374.50 44.53 34
10	347.26 54.92 23	377.68 62.48 22	336.38 42.48 21	327.25 37.95 20	313.65 40.93 20
11	501.00 40.80 54	491.04 41.78 53	483.25 31.89 51	484.00 37.67 48	477.70 42.01 47

Table 4.4. The simulation results for system 2

Parts \ $(1/X_1, X_2)$	(1.0,1.0)	(1.1,1.0)	(1.2,1.0)	(1.3,1.0)	(1.4,1.0)	(1.5,1.0)
1	294.68 42.13 161	229.79 35.31 145	214.35 32.56 135	203.56 34.81 124	190.20 27.75 114	186.99 24.56 107
2	335.59 38.13 140	244.48 33.00 128	233.79 33.38 117	224.70 24.25 107	219.52 24.75 102	210.02 22.44 94
3	418.82 44.75 110	311.52 35.75 100	288.41 37.75 93	273.78 26.69 85	271.65 30.56 79	263.84 27.06 74
4	588.49 55.50 80	424.04 46.25 73	406.46 47.81 67	387.68 34.50 62	376.47 28.81 58	376.92 37.94 53
5	810.21 71.31 90	606.95 70.50 82	563.05 64.13 75	541.52 46.56 70	526.27 46.13 64	510.05 33.38 60
6	797.82 66.19 101	587.11 54.25 91	564.56 57.19 85	537.64 51.44 78	519.53 44.94 70	509.78 41.56 67
7	891.84 67.00 70	647.78 56.56 63	622.14 53.63 58	589.89 39.56 54	581.14 56.44 50	554.66 47.06 47
8	976.02 76.31 66	745.85 69.81 60	727.44 61.69 54	676.67 68.56 51	648.91 57.13 47	614.70 35.38 44

Table 4.4. (continued)

Parts \ $(1/X_1, X_2)$	(1.0,1.1)	(1.1,1.1)	(1.2,1.1)	(1.3,1.1)	(1.0,1.2)	(1.1,1.2)
1	213.33 32.81 162	190.44 28.25 147	188.24 27.31 134	176.13 22.81 124	190.53 29.19 161	176.80 26.75 146
2	229.64 29.25 140	214.83 28.63 128	206.20 23.19 117	199.34 19.44 107	202.50 24.44 139	195.26 22.25 128
3	289.53 37.44 110	270.78 28.69 101	256.01 26.38 93	241.58 20.69 85	258.96 28.44 111	240.70 21.13 100
4	405.42 43.56 80	375.79 34.94 73	356.85 32.19 66	353.02 32.88 52	354.02 32.00 81	354.82 33.94 73
5	580.80 56.44 91	523.02 42.88 81	493.01 34.56 75	480.77 26.44 70	505.50 42.38 80	473.45 37.44 82
6	555.44 48.25 100	509.77 59.74 91	499.76 44.31 83	478.36 33.44 76	492.32 44.81 99	480.79 45.56 92
7	621.23 59.38 71	566.19 49.25 63	537.34 45.81 59	525.13 45.56 54	539.40 44.75 70	522.63 48.38 63
8	714.97 62.81 66	648.32 52.63 60	618.35 55.00 55	587.32 39.00 50	628.15 48.00 66	608.68 43.50 60

Table 4.4. (continued)

$(1/X_1, X_2)$ Parts	(1.2,1.2)	(1.0,1.3)	(1.1,1.3)	(1.0,1.4)	(1.0,1.5)
1	173.96 21.63 134	168.62 23.31 160	162.36 20.56 147	152.30 19.25 161	149.51 15.41 161
2	188.55 18.56 117	152.01 19.25 141	181.20 20.94 127	170.39 15.99 141	163.81 14.92 140
3	234.73 20.56 91	234.38 22.56 111	225.97 20.38 101	216.41 20.81 111	202.55 17.88 110
4	327.97 24.38 66	327.86 27.13 80	314.62 26.25 73	304.22 22.13 80	286.60 20.63 81
5	459.20 28.75 75	451.61 37.06 90	433.73 29.50 82	416.99 21.06 80	389.88 22.13 90
6	453.90 36.69 82	453.00 40.63 100	434.96 34.44 91	415.18 32.13 102	391.14 29.31 99
7	506.69 46.31 59	498.86 40.38 70	479.84 34.69 64	451.56 29.56 70	432.37 34.06 71
8	562.75 33.94 55	567.32 39.81 66	541.52 44.44 60	516.20 39.44 66	486.92 33.19 66

From Table 4.3 and Table 4.4 it can be seen that the mean of production lead time of each part decreases at a faster rate at the beginning and at a slower rate later as $1/X_1$ increases at a given X_2 or vice versa. In Table 4.4 the response of the lead time seems to be more sensitive to the changes of X_2 than that of $1/X_1$. The standard deviation of the lead time of each part also decreases as $1/X_1$ increases at a given X_2 or vice versa. However, its change is more irregular compared to the mean.

4. Regression analysis and results

Many different types of curves were tested to find the best one by fitting curves to the data obtained from the simulation study. Based on the results from the test fittings, the following curves are selected as the empirical functional relations of interest.

System 1: $\ln \mu_{LT} = a_0 + a_1X_1 + a_2X_1^2$

$$\ln \sigma_{LT}^2 = b^0 + b_1 \frac{1}{X_1}$$

System 2: $\ln \mu_{LT} = C_0 + C_1X_1 + C_2 \frac{1}{X_2} + C_3X_1^2 + C_4 \frac{1}{X_2^2} + C_5 \frac{X_1}{X_2}$

$$\ln \sigma_{LT}^2 = d_0 + d_1 \frac{1}{X_1} + d_2X_2$$

where $a_0, \ldots, d_2$ are regression coefficients and μ_{LT} and σ_{LT} are the mean and variance of the production lead time of each part respectively.

In estimating the regression coefficients, for both systems the regression on the variance is weighted by the sample size of each part. However, the regression on the mean is weighted only for system 2 by the sample variance of each part.

Generally speaking, the fittings are reasonably good. The plots of residuals against X_1 at a given X_2 and X_2 at a given X_1 are flat and there is no noticeable trend. Also the magnitude of the variance of residuals of each part is comparable to each other. The value of R-square is bigger than 90% for most of the parts and this justifies good fittings.

The estimated regression coefficients are presented in Tables 4.5 and 4.6. For each part the first row shows the coefficients of mean and the second row shows those of variance.

Figures 4.3 and 4.4 show the sample mean and its regression line, and the sample variance and its regression line respectively for parts 1 and 2 of system 1.

Figure 4.5 shows the sample mean and its regression line for part 6 of system 2 when $1/X_1$ changes with $X_2 = 1.0$ and when X_2 changes with $1/X_1 = 1.0$.

Figure 4.6 shows the sample variance and its regression line for part 6 of system 2 when $1/X_1$ changes with $X_2 = 1.0$ and when X_2 changes with $1/X_1 = 1.0$.

Table 4.5. The estimated regression coefficients for system 1

Coefficients / Parts	a_0, a_1, a_2 / b_0, b_1		
1	7.57262 9.22089	-6.56323 -1.52527	4.83975
2	8.95661 9.49944	-8.60696 -1.62078	5.87359
3	8.64820 10.35096	-7.63679 -1.95263	5.37906
4	6.12187 10.65452	-0.95406 -2.25278	1.28850
5	7.56174 10.0041	-3.83496 -1.44604	3.03228
6	7.10126 9.28840	-2.67679 -1.76790	1.88665
7	7.82413 11.31598	-5.91787 -2.42913	4.64869
8	7.65484 9.41209	-4.44339 -1.41525	3.22596
9	6.76243 9.36623	-3.14228 -1.02779	2.85753
10	6.73226 8.60464	-3.33257 -0.60178	2.86792
11	7.76452 9.25840	-4.63841 -1.38362	3.39507

Table 4.6. The estimated regression coefficients for system 2

Coefficients / Parts	$c_0, c_1, c_2, c_3, c_4, c_5$ d_0, d_1, d_2					
1	15.77057 13.33526	-12.83606 - 2.04304	-13.11142 - 3.86646	3.41709	3.99692	8.42583
2	17.84969 13.31772	-15.20100 - 2.20303	-15.10459 - 3.90060	4.20397	4.61533	9.40756
3	16.86170 13.58264	-14.81526 - 2.29820	-12.95179 - 3.82346	4.47626	3.81567	8.61164
4	18.58838 13.93402	-16.47582 - 2.03590	-14.21390 - 4.01708	4.96530	4.09702	9.36883
5	16.88387 17.41273	-14.22046 - 3.31800	-12.13968 - 5.46836	4.24436	3.47859	8.42073
6	18.16870 13.30838	-15.17553 - 1.80347	-13.83052 - 1.22319	4.37391	4.03700	9.06320
7	19.60497 12.41749	-16.73217 - 1.25870	-15.32274 - 2.89126	4.80386	4.43318	9.97074
8	13.73197 14.56079	- 9.55318 - 2.28710	- 9.25442 - 3.62400	2.60122	2.83992	6.48355

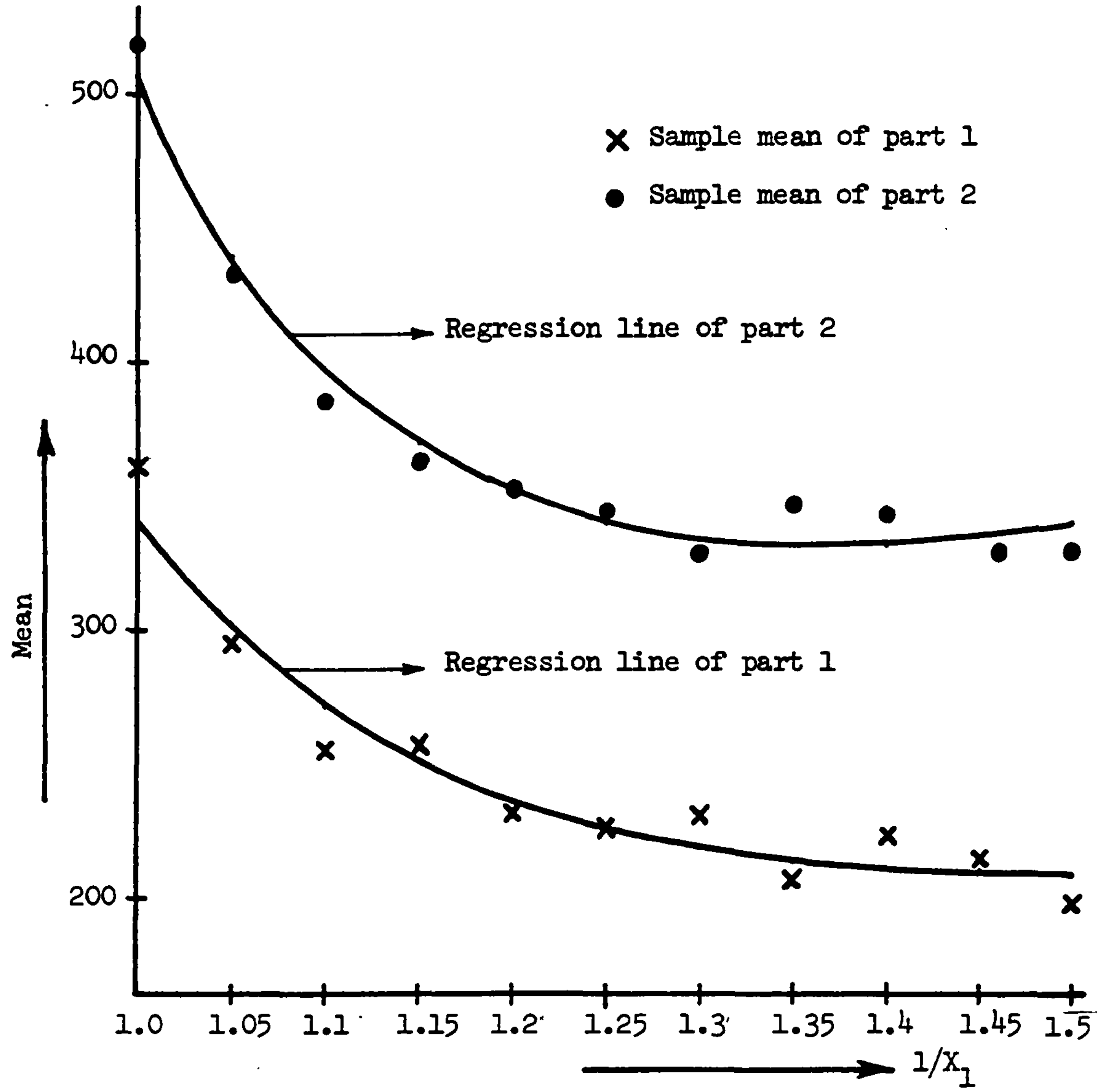


Figure 4.3. The sample mean and its regression line for part 1 and 2 of system 1

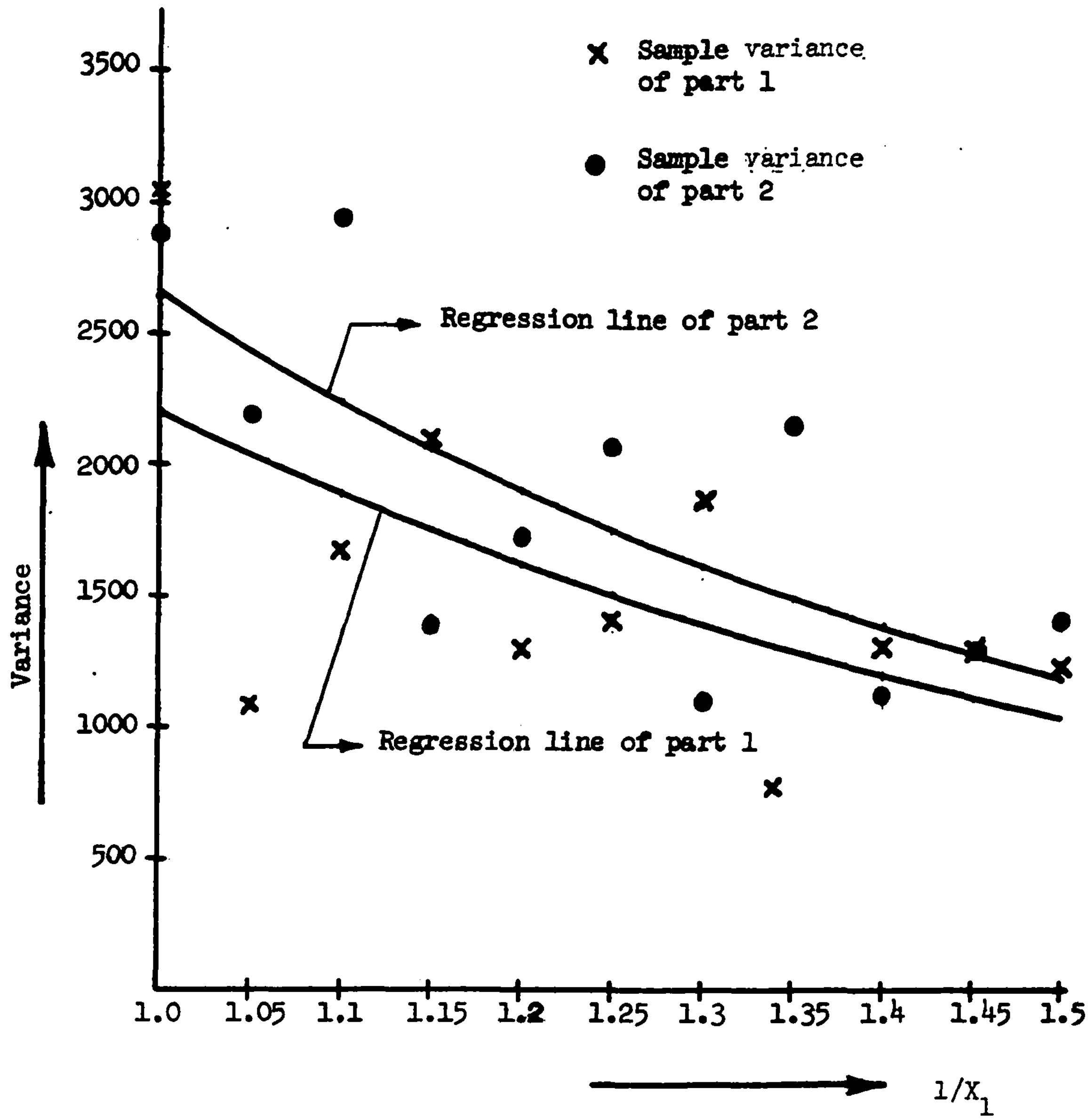


Figure 4.4. The sample variance and its regression line for parts 1 and 2 of system 1

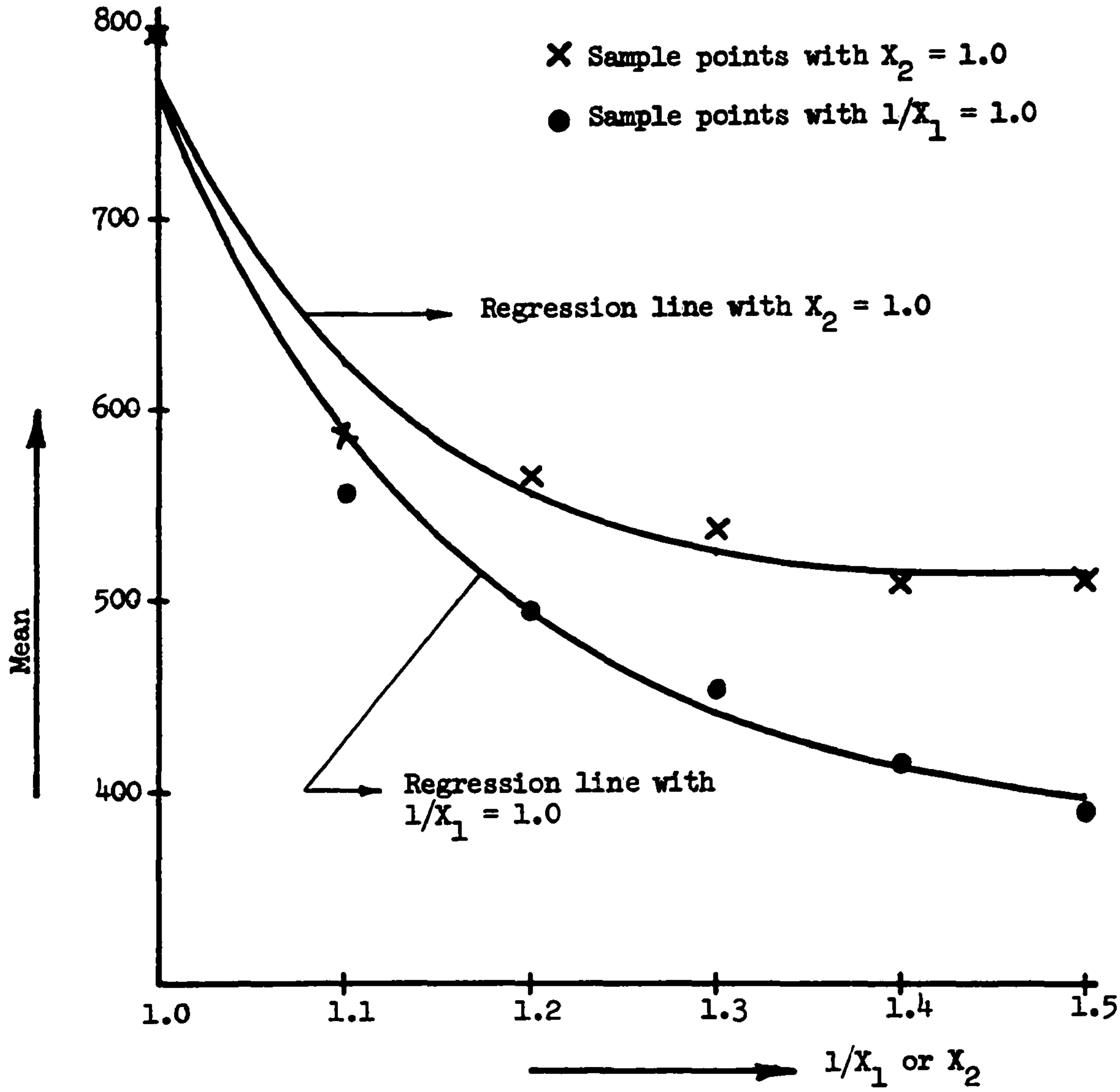


Figure 4.5. The sample mean and its regression line for part 6 of system 2

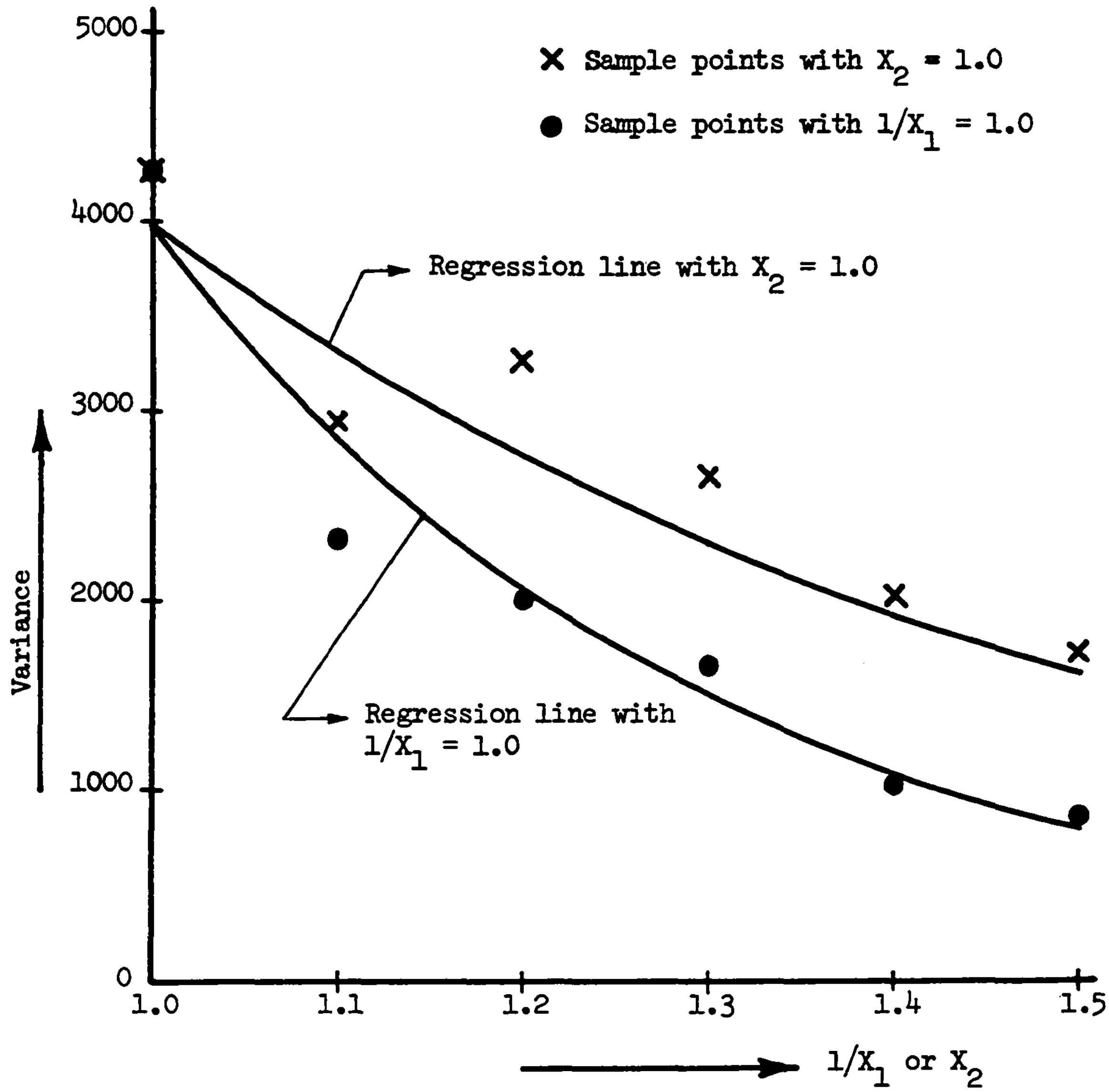


Figure 4.6. The sample variance and its regression line for part 6 of system 2

5. Normality assumption for lead time demand

Although the mean and variance of the production lead time of each part and their functional relations with X_1 and X_2 were obtained in previous sections, the distribution of production time was not investigated thoroughly. As a crude method of finding the distribution, five sets of samples were drawn randomly and their histograms were constructed. Three of them were reasonably close to normal. But two of them were a little bit skewed to the left. Leaving the actual distribution of the production lead time as a further research, it will be assumed to be normal. Stanley (1968) investigated the production lead times of multiple parts in a shop which is very similar to the two hypothesized systems of this chapter. He found that the production lead times were normal. His finding is another justification for the above assumption.

Danish (1972) studied the distribution of lead time demand for various combinations of different lead time distributions and demand distributions. One of the combinations was that the distribution of the demand was Poisson and that of lead time was normal, which is the case of interest in this section. He concluded that the distribution of lead time demand was normal. In arriving at his conclusion, he discretized normal distribution by using an interval $t - 1/2$ and $t + 1/2$ for $t=1,2,\ldots,m$ where m lies beyond the $\mu_{LT} + 3\,\sigma_{LT}$. Then he expressed the probability function of lead time demand as a convolution of two probability functions of discrete random variables. He plotted the convoluted probability function by means of a computer program for arbitrary values of μ_{LT}, σ_{LT} and λ which is the mean of the Poisson distribution. The plot was well matched with the theoretical normal distribution with mean $\lambda \cdot \mu_{LT}$ and variance $\lambda \cdot \sigma_{LT} + \lambda^2 \cdot \sigma_{LT}^2$. Danish's conclusion is the justification for the normality assumption for lead time demand in systems 1 and 2.

6. Optimum trade-off among WIP inventory, the number of production orders and service rates

It has been seen that the mean and variance of production lead time of each part decrease as $1/X_1$ increases (or X_1 decreases) at a given X_2, or X_2 increases at a given $1/X_1$, or both increase. As the mean of each part decreases, so does its average WIP in production. Particularly, when X_1 decreases the average WIP in production decreases due to the decrement of mean as well as the decrement of yearly demand of each part (see equation 4.1). Similarly as the mean and variance of each part decrease, so does its average WIP inventory in storage. By looking at equation 4.9, this is not obvious. However the average WIP inventory of each part was calculated from equations 4.9, 4.12 and 4.13 using a FORTRAN program and its decrease was indicated. Equations 4.12 and 4.13 were used in the calculation because of the normality assumption for lead time demand which has been discussed in the previous section.

The resulting value of $B(Q,r)$ for each part was approximately one percent or less of the average WIP inventory in storage. Also the value of the second term in equation 4.12 was almost negligible.

Consequently the average WIP inventory of each part could be calculated from equations 4.3 and 4.8 of the approximate treatment and the resulting error would be negligible for practical purposes.

Figure 4.7 shows the total average WIP inventory in production of all 11 parts of system 1 and its counterpart in storage as X_1 decreases from 1.0 to 0.5. Note that the decrease of X_1 from 1.0 to 0.5 is equivalent to the increase of $1/X_1$ from 1.0 to 2.0.

Figure 4.8 shows the total average WIP inventory in production of all 8 parts of system 2 and its counterpart in storage as X_1 decreases from 1.0 to 0.5 with $X_2 = 1.0$ and as X_2 increases from 1.0 to 1.5 with $X_1 = 1.0$.

It is possible to draw a family of curves similar to those in Figures 4.7 and 4.8 by changing X_1 at various values of X_2 and vice versa. In fact all the possible curves would compose WIP inventory surfaces in three dimensional space. However these surfaces have not been obtained in this research.

In figures 4.7 and 4.8 the response of the WIP inventory in production is much more sensitive than that of the WIP inventory in storage. Also the average inventory level in production is much bigger than that in storage, especially when the value of X_1 or X_2 is near 1.0.

If the decrease of X_1 and the increase of X_2 incur some costs, it is possible to locate an optimum trade-off point among the total WIP inventory (the total in production plus the total in

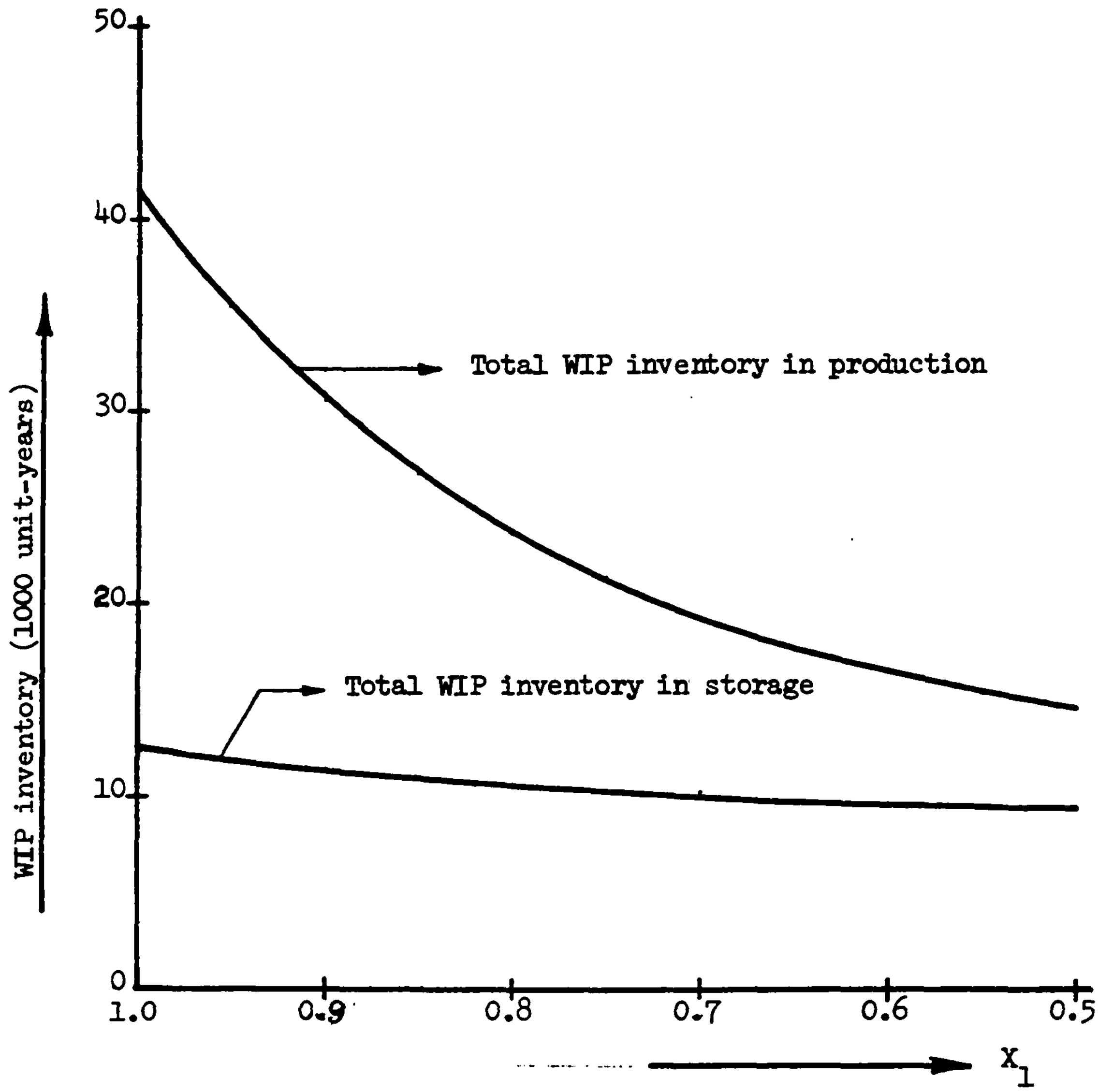


Figure 4.7. Total WIP inventory in production and storage

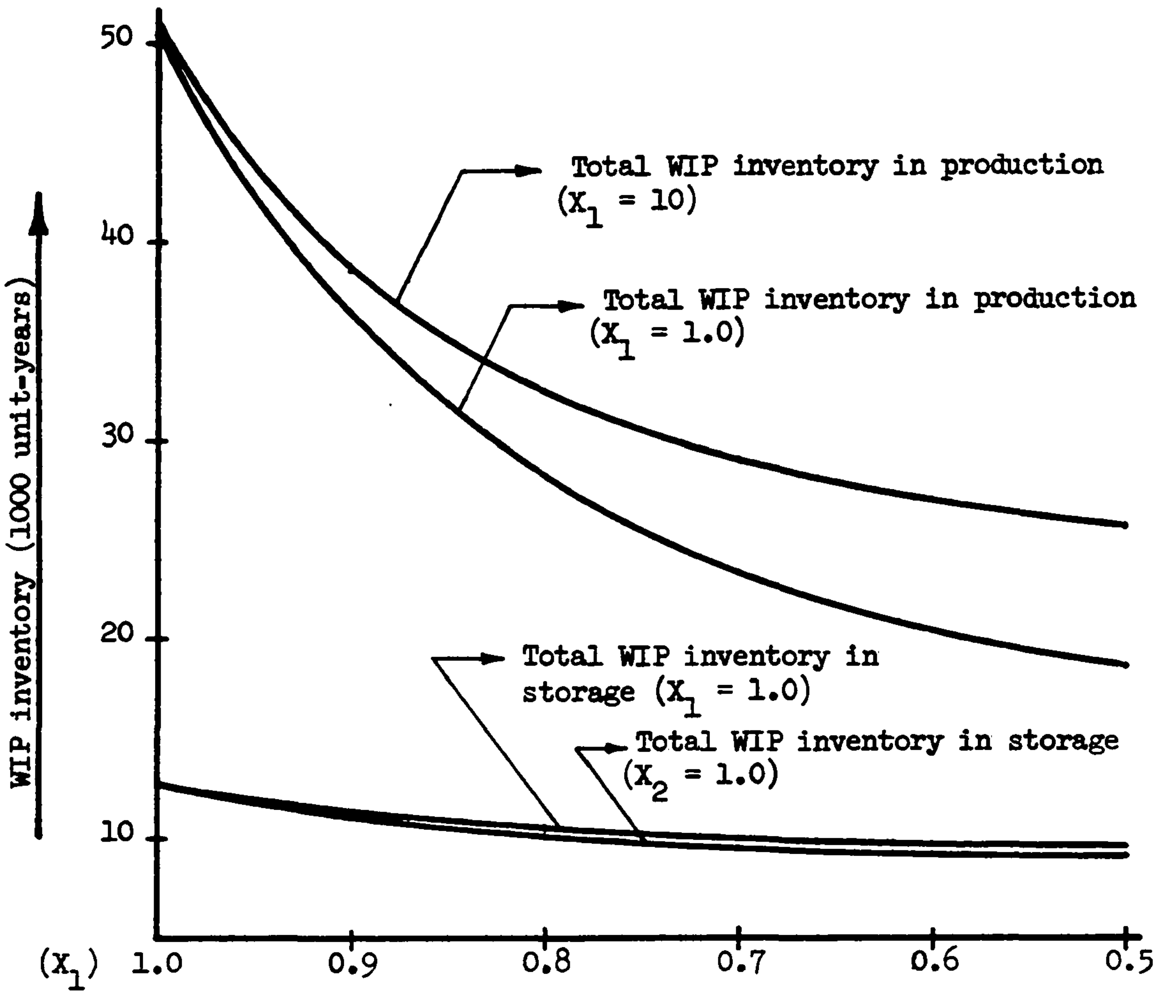


Figure 4.8. Total WIP inventory in production and storage in system 2

storage), X_1 and X_2, which minimizes inventory holding cost plus the costs associated with X_1 and X_2. By providing the following fictitious data the optimum points for systems 1 and 2 are obtained.

System 1: $V_{initial}$ for all parts = \$30/unit
V_{final} for all parts = \$60/unit
I (inventory carrying charge) = 0.2
The cost for decreasing X_1 = \$5,700/one percent

System 2: $V_{initial}$ for all parts = \$30/unit
V_{final} for all parts = \$60/unit
I (inventory carrying charge) = 0.2
The cost for decreasing X_1 = \$6,000/one percent
The cost for increasing X_2 = \$6000/one percent

Different values could be assigned to $V_{initial}$, V_{final} and I for each part of the systems. For simplicity the same value is assigned to each of them for each part. For system 2 two optimum points are located; the optimum value of X_1 with $X_2 = 1.0$ and the optimum value of X_2 with $X_1 = 1.0$. It is more desirable to locate the optimum pair of X_1 and X_2 when both of them change. However this has not been done.

Figures 4.9 and 4.10 show the inventory holding cost plus the costs associated with X_1 and X_2 and optimum points for system 1 and for system 2 respectively.

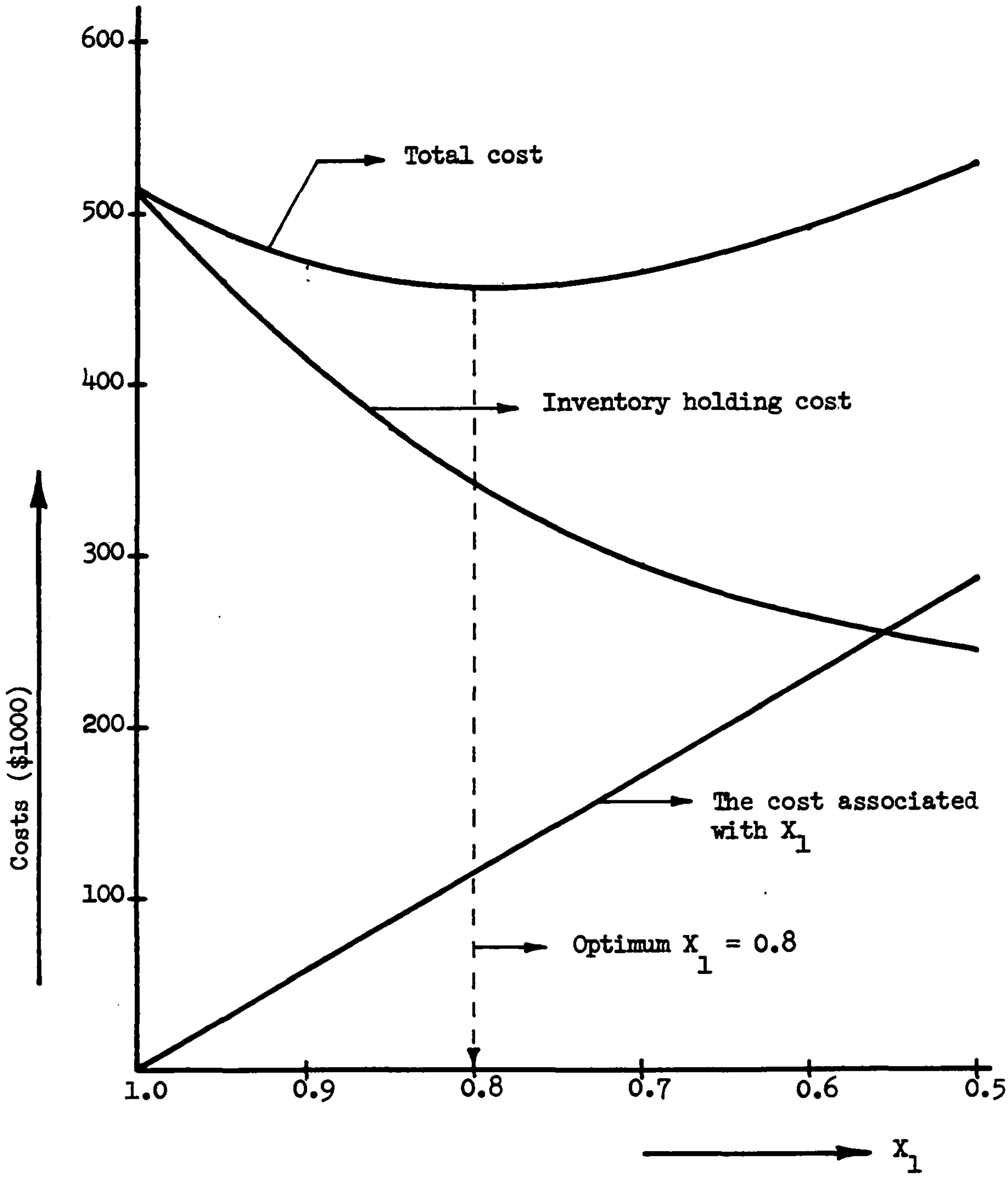


Figure 4.9. Total cost curve for system 1

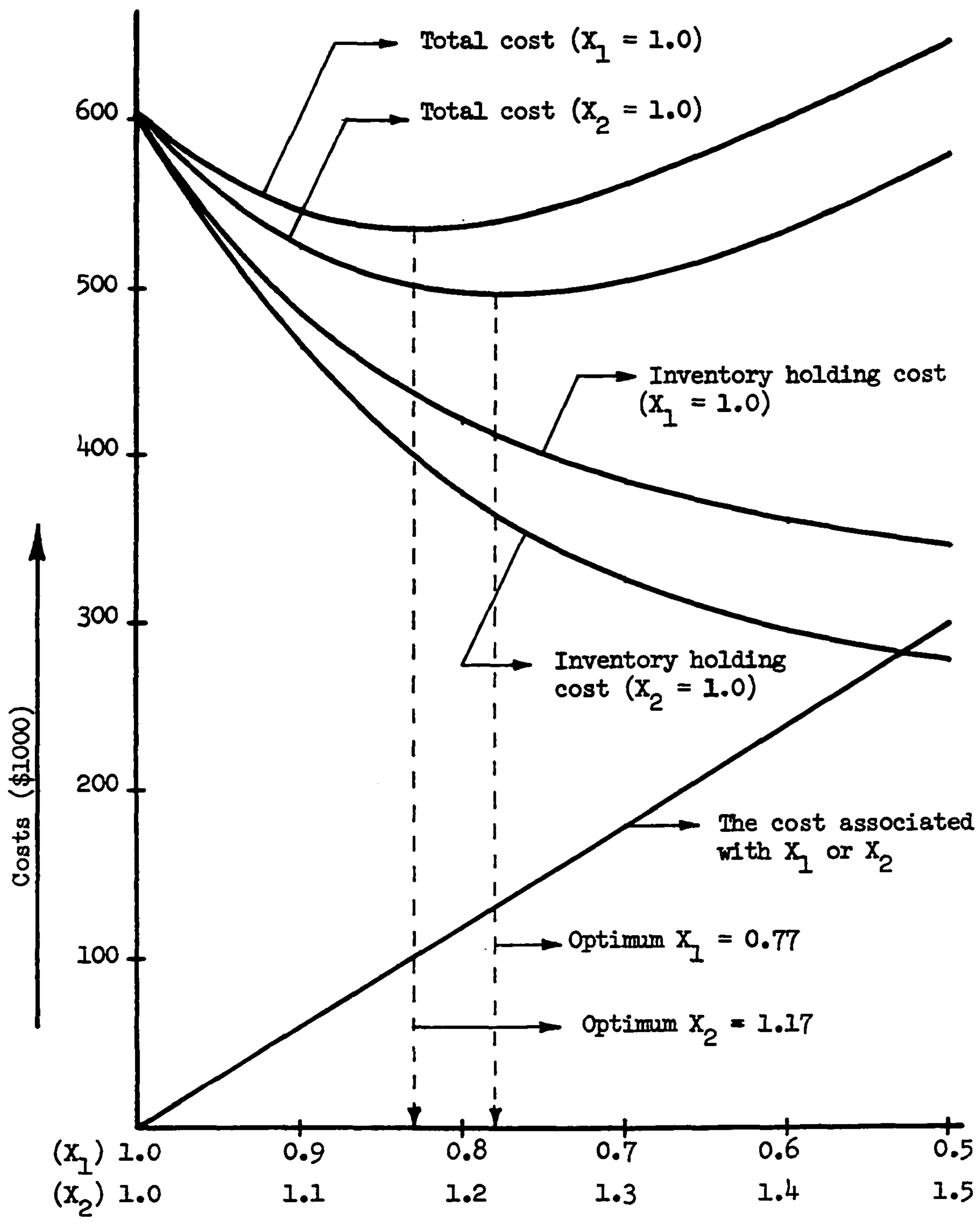


Figure 4.10. Total cost curve for system 2

V. CONCLUSIONS AND RECOMMENDATIONS

A. Deterministic Case

Although the model developed in Chapter III combines the decision on inventory and the decision on scheduling, it is limited to a case where the following conditions are met: (1) the total number of different parts is large, (2) the yearly demand of each part is large, (3) the yearly production cycle of each machine center is bounded by an upper limit and restricted to an integer, (4) the dollar value of each part at any production stage is known, and (5) the available machine hours are the only constraint to production scheduling. In addition, the model involves approximation and the application of the model requires a tedious computation routine which could be computerized. Hence when one wants to use the model, its strong and weak points should be evaluated.

Above all, the real intention of developing the model is to calculate the WIP inventory of a multi-stage inventory/production system by using the concept of cumulative production and demand. The concept is workable even though improvement for the application of the concept is desirable at this stage.

The model is used as a vehicle to link the problems of inventory and machine replacement decisions in the later part of Chapter III. A simplifying assumption which could be challenged (Section D-2) virtually ignores the basic concept involved in the earlier development.

There should be a better way to link these problems without defying the basic concept.

Generally speaking, the amount of WIP inventory between machine center i and machine center $i + 1$ decreases as N_i and N_{i+1} increase simultaneously. However, this will increase the number of setups at both machine centers and the total production cost will be increased. An alternative to avoid this situation is to make N_{i+1} an integer multiple of N_i. Then α_i will be 1 and the minimum distance to be used in calculating the installation inventory will be the full distance between the demand line and one apex of the actual roduction line of machine center $i + 1$. But the full distance depends on N_{i+1} and the savings in inventory holding cost by making N_{i+1} an integer multiple of N_i depends on the magnitude of N_{i+1}. Also the change on N_{i+1} will affect the inventory between machine center $i + 1$ and machine center $i + 2$. In terms of the total production cost which includes the setup cost as one component of it, the effect of making N_{i+1} an integer multiple of N_i on the total production is a complicated one. Because of this reason making the production cycle of machine center $i + 1$ an integer multiple of that of machine center i for $i = 2, 3, \ldots, M$ does not guarantee the minimization of the total production cost. The only way to find an optimum set of $N_1, N_2, \ldots, N_M$ is to go through an efficient enumeration procedure like Dynamic programming or Branch and Bound technique.

As mentioned before there is big room for improvement and further research in the model. The following represents improvements and further research necessary in the model:

(1) It is desirable to take into account the raw material inventory and finished product inventory in the model. The consideration of these two inventories will increase the number of decision variables and will not change the model materially.

(2) In the model the number of containers used at each machine center is treated as a dependent variable. No cost reflecting the economy of scale is assigned to this variable. The model would be more realistic if the number of containers was considered another independent decision varaible with an associated cost.

(3) It is desirable to computerize the computational routine of the branch and bound algorithm. Also the calculation of the adverse minimum for machine replacement can be done by computer.

(4) The inventory which is tied up in transit is ignored in the model because it is constant. However, it is possible to include that in the model as another independent decision variable. Since the amount of the

inventory tied up in transit is significant in the real world, the inclusion should be investigated.

(5) When replacing a machine, it is assumed that all the machines at a given machine center are replaced at the same point in time. However, it seems more logical to replace x old machines with y new machines where x and y are non-negative integers. There is theoretical difficulty associated with this generalization.

(6) The planning horizon of the model is infinite. Attention should be given to the case where the planning horizon is finite.

(7) It might be very interesting attempt to apply the concept of cumulative production and demand to the case where the rate of production and the rate of demand are both random variables having known distributions.

B. Stochastic Case

While the WIP inventory of the deterministic case depends on the production cycle of each machine center, it depends on the production lead time of each part in the stochastic case. There is a definite functional relationship between the level of congestion in the production floor and the production lead time. The mean and variance of the production lead time decreases rapidly at the beginning and then the rate of decrement declines as the level of congestion goes down. This relationship is used to find an optimum level of congestion in terms of minimizing inventory holding cost and penalty cost for lowering the level of congestion.

The amount of WIP inventory tied up in the production floor is much larger than that in storage. Also, the response of the average level of inventory to the level of congestion is much more sensitive for the WIP inventory in the production floor than that in storage. This means that a sizable savings will occur in the WIP inventory in the production floor by decreasing the level of congestion.

In the study of the stochastic production system identifying the functional relationship between the level of congestion and the production lead time is the prime objective. However, establishing a procedure to find an optimum level of congestion is another main objective. The procedure used in Chapter IV is only one way to find an optimum level of congestion. Other methods which could be more efficient should be investigated.

As in the deterministic case, there are many questions left unanswered and areas which need to be investigated further.

(1) It is desirable to consider both the raw material inventory and finished product inventory in finding an optimum level of congestion and WIP inventory.

(2) The batch size is predetermined in the model to simplify

the situation. However, the batch size is one important decision variable and hence its effect on the production lead time should be identified.

(3) The distribution of production lead time as well as the distribution of lead time demand should be identified.

(4) Most manufacturing companies operate on multiple shifts. The model would be more realistic if the aspect of multiple shifts was included in the computer simulation.

(5) A complete picture of the level of the WIP inventory as a function of X_1 and X_2 has not been obtained. Also a complete picture of the inventory holding cost has not been acquired.

(6) It is possible to establish optimum machine replacement policy by assuming that the available machine hours decrease via more frequent machine breakdowns as the machine gets older.

(7) The results of Jackson's paper (1957) can be used to develop an analytical functional relationship between the level of congestion and the production lead time. The effect of the batch size of each part on the production lead time can be identified analytically. Hence the batch size can be handled as an independent

decision variable in developing a cost model. Also it is possible to establish optimum machine replacement policy based on an extended cost model. Such development will be important to clarifying questions and problems raised in this research.

VI. BIBLIOGRAPHY

Anderson, D. R. 1968. Transient and steady-state minimum cost in-process inventory capacities for production lines. Unpublished Ph.D. thesis. Purdue University.

Ancker, C. J., Jr., and A. V. Gafarian. 1961. Queueing with multiple Poission inputs and exponential service times. Operations Research 3, No. 2:321-327.

Bell, J. C. 1973. Inventory as a dynamic element in company performance. Proceedings of Summer Simulation Conference, (July), Montreal, Quebec, 979-983.

Bramson, M. J. 1962. The variable lead-time problem in inventory control, A survey of the literature--Part 1. Operations Research Quarterly 13, No. 1:41-53.

Burgin, T. A. 1972. Inventory control with normal demand and Gamma lead times. Operations Research Quarterly 23, No. 1:73-80.

Burke, P. J. 1956. The output of a queueing system. Operations Research 4, No. 4:699-704.

Buzacott, J. A. 1971. The role of inventory ranks in flow-line production systems. International Journal of Production Research 9, No. 4:425-436.

Clark, A. J., and H. Scarf. 1960. Optimal policies for a multi-echelon inventory problem. Management Science 6, No. 4:475-490.

Clark, C. E., and A. J. Rowe. 1960. Inventory policies and related numerical approximations. The Journal of Industrial Engineering 11, No. 1:8-17.

Conway, R. W., W. L. Maxwell, and L. W. Miller. 1967. Theory of Scheduling. Addison-Wesley, Reading, Massachusetts.

Crowston, W. B., M. Wagner, and J. F. Williams. 1973. Economic lot size determination in multi-stage assembly systems. Management Science 19, No. 5:517-527.

Danish, A. A. 1972. A note on the problem of calculating the re-order point in a stochastic inventory problem. Production and Inventory Management 13, No. 3:11-32.

Das, C. 1975. Effect of lead time on inventory: A static analysis. Operational Research Quarterly 26, No. 2:273-282.

Disney, R. L. 1975. Random flow in queueing networks: A review and critique. AIIE Transactions 8, No. 3:268-288.

Ekey, D. C., J. B. Talbird, and T. L. Newberry. 1961. Inventory reorder points for conditions of variable demand and lead time. The Journal of Industrial Engineering 12, No. 1:32-34.

Fairhurst, J. H. 1973. An empirical attempt to relate stock lead-time variability to shop loading levels: A case study. International Journal of Production Research 11, No. 3:205-217.

Gross, D., and A. Soriano. 1969. The effect of reducing leadtime on inventory levels--Simulation analysis. Management Science 16, No. 2:B61-B76.

Hadley, G., and T. M. Whitin. 1963. Analysis of Inventory Systems. Prentice-Hall, Englewood Cliffs, N. J.

Hollier, R. H. 1964. Two studies of work flow control. International Journal of Production Research 3, No. 4:253-283.

Jackson, J. R. 1957. Networks of waiting lines. Operations Research 5, No. 4:518-521.

Jackson, J. R. 1963. Jobshop-like queueing systems. Management Science 10, No. 1:131-142.

Koenigsberg, E. 1959. Production lines and internal storage--A review. Management Science 5, No. 4:410-433.

Kotiah, T. C. T., and N. B. Slater. 1973. On two-server Poission queues with two types of customers. Operations Research 21, No. 2:597-603.

Kuester, J. L., and J. H. Mize. 1973. Optimization Techniques with FORTRAN. McGraw-Hill, New York, N. Y.

Land, A., and A. Doig. 1960. An automatic method of solving discrete programming problems. Econometrica 28, No. 3:497-520.

McRoberts, K. L., and S. H. Chung. 1975. Technical data relating to the small manufacturing industry in Iowa, ERI Project 2014, Department of Industrial Engineering and Engineering Research Institute, Iowa State University.

Plossl, G. W. 1971. How much inventory is enough? Production and Inventory Management 12, No. 2:1-22.

Plossl, G. W. 1975. Personal correspondence.

Plossl, G. W., and O. W. Wight. 1973. Capacity planning and control. Production and Inventory Management 14, No. 3:31-67.

Rosenshine, M. 1975. Queueing theory: The state-of-the-art. AIIE Transactions 7, No. 3:257-267.

Schriber, T. J. 1974. Simulation Using GPSS. John Wiley & Sons, New York, N. Y.

Schwarz, L. B., and L. Schrage. 1975. Optimal and system myopic policies for multi-echelon production/inventory assembly systems. Management Science 21, No. 11:1283-1294.

Shamma, M. M., D. R. Anderson, and B. D. Sellers, 1973. Simulation of sequential production systems with in-process inventory. Proceedings of Winter Simulation Conference, (January), San Francisco, California, 85-92.

Shore, B. 1975. Replacement decisions under capital budgeting constraints. The Engineering Economist 20, No. 4:243-256.

Silver, E. A. 1970. A modified formula for calculating customer service under continuous inventory review. AIIE Transactions 2, No. 3:241-245.

Simpson, K. F., Jr. 1958. In-process inventories. Operations Research 6, No. 6:863-873.

Smith, G. W. 1973. Engineering Economy: Analysis of Capital Expenditure. 2nd edition. The Iowa State University Press, Ames, Iowa.

Stanley, G. J. 1968. On determining capacity levels at machine centers in a job shop manufacturing system. Unpublished Ph.D. thesis. Oklahoma State University.

Taha, H. A., and R. W. Skeith. 1970. The economic lot sizes in multistage production systems. AIIE Transactions 2, No. 2: 157-162.

Terborgh, G. 1949. Dynamic Equipment Policy. McGraw-Hill, New York, N. Y.

Wight, O. 1970. Input/output control, A real handle on lead time. Production and Inventory Management 11, No. 3:9-31.

VII. ACKNOWLEDGMENTS

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VIII. APPENDIX A: THE VALIDITY OF THE DEVELOPMENT IN SECTION III-A-6

In Figure 3.12 the actual production line of machine center $i + 1$ should be completely covered by the available production line at machine center $i + 1$ after the latter has been shifted to the right horizontally by EF. However, it may be possible that a certain portion of the available production line is crossed over by the actual production line by the shift. This situation can occur if

$$n_i < \frac{\beta_{i+1}}{2(P_i - D)} \cdot \frac{D(P_{i+1} - P_i)}{P_{i+1}} .$$

Figure A.1 shows the pseudo available and actual production lines of machine center $i + 1$ in Figure 3.12 before the latter is shifted by CG. In Figure A.1, $L_i = \frac{1}{N_i} = t \cdot \beta_{i+1}$ and $L_{i+1} = \frac{1}{N_{i+1}} = t \cdot \alpha_i$ for some t $(0 < t < 1)$. $C_1, C_2, \ldots, C_{\beta_{i+1}}$ are such horizontal distances that $0 \leq C_1 = L_{i+1} - \ell_1 . L_i < L_i$, $0 \leq C_2 = 2.L_{i+1} - \ell_2 . L_i < L_i, \ldots,$ $0 \leq C_{\beta_{i+1}-1} = (\beta_{i+1} - 1)L_{i+1} - \ell_{\beta_{i+1}-1} . L_i < L_i$, $C_{\beta_{i+1}} = \beta_{i+1} . L_{i+1} - \ell_{\beta_{i+1}} . L_i = 0$ where $\ell_1, \ell_2, \ldots, \ell_{\beta_{i+1}} \varepsilon (1,2,\ldots, \alpha_i)$ and $\ell_1 \leq \ell_2 \leq \cdots \leq \ell_{\beta_{i+1}-1} \leq \ell_{\beta_{i+1}} = \alpha_i$. The place which should be examined for the crossing due to the rightward shift by EF in Figure 3.12 is where the value of C_i is minimum.

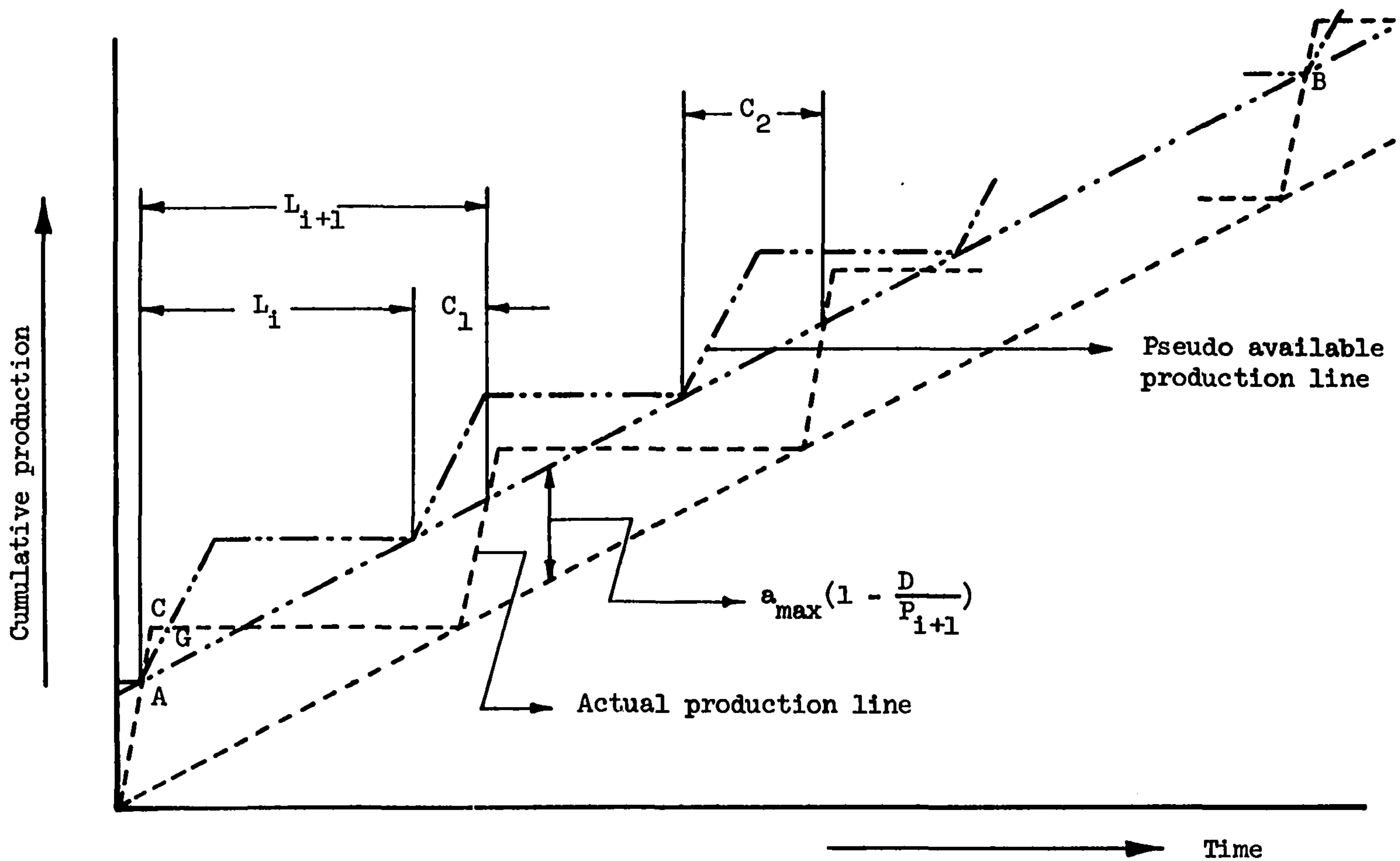

Figure A.1. Pseudo available production line and actual production line of machine center i+1 when $P_i < P_{i+1}$

Since $L_i = t \cdot \beta_{i+1}$ and $L_{i+1} = t \cdot \alpha_i$, $\frac{C_1}{t} = \alpha_i - \ell_1 \cdot \beta_{i+1}$,

$\frac{C_2}{t} = 2.\alpha_i - \ell_2 \cdot \beta_{i+1}, \ldots, \frac{C_{\beta_{i+1}-1}}{t} = (\beta_{i+1} - 1) \cdot \alpha_i - \ell_{\beta_{i+1}-1} \cdot \beta_{i+1}$,

$\frac{C_{\beta_{i+1}}}{t} = 0$. Because α_i and β_{i+1} are relatively prime integers,

$\frac{C_1}{t}, \frac{C_2}{t}, \ldots, \frac{C_{\beta_{i+1}}}{t}$ will take a unique value from $(0,1,2,\ldots, \beta_{i+1}-1)$.

Then the minimum, $\frac{C_{min}}{t}$, of them is 1 and $C_{min} = \frac{1}{N_i \cdot \beta_{i+1}}$

$= \frac{1}{N_{i+1} \cdot \alpha_i}$.

Figure A.2 shows a portion of Figure A.1 where the value of C_i is minimum. In Figure A.2, V is one point of the available production line at machine center $i + 1$, which can cross over the actual production line at W. The distance between V and W is

$C_{min} (1 + \frac{D}{P_{i+1}} - \frac{2D}{P_i}) + EF'$ where

$EF' = \{\frac{2D}{N_{i+1}} \cdot \frac{1}{\alpha_i} - K_i' \cdot \frac{D}{N_i} \cdot \frac{1}{n_i}\} (\frac{1}{P_i} - \frac{1}{P_{i+1}})$ and K_i' is the

integer part of $(\frac{2 \cdot n_i}{\beta_{i+1}})$. When $(\frac{2 \cdot n_i}{\beta_{i+1}})$ is integer itself,

$K_i' = (\frac{2 \cdot n_i}{\beta_{i+1}}) - 1$. After the available production line has been

shifted leftward by CG in Figure 3.12, the total distance between

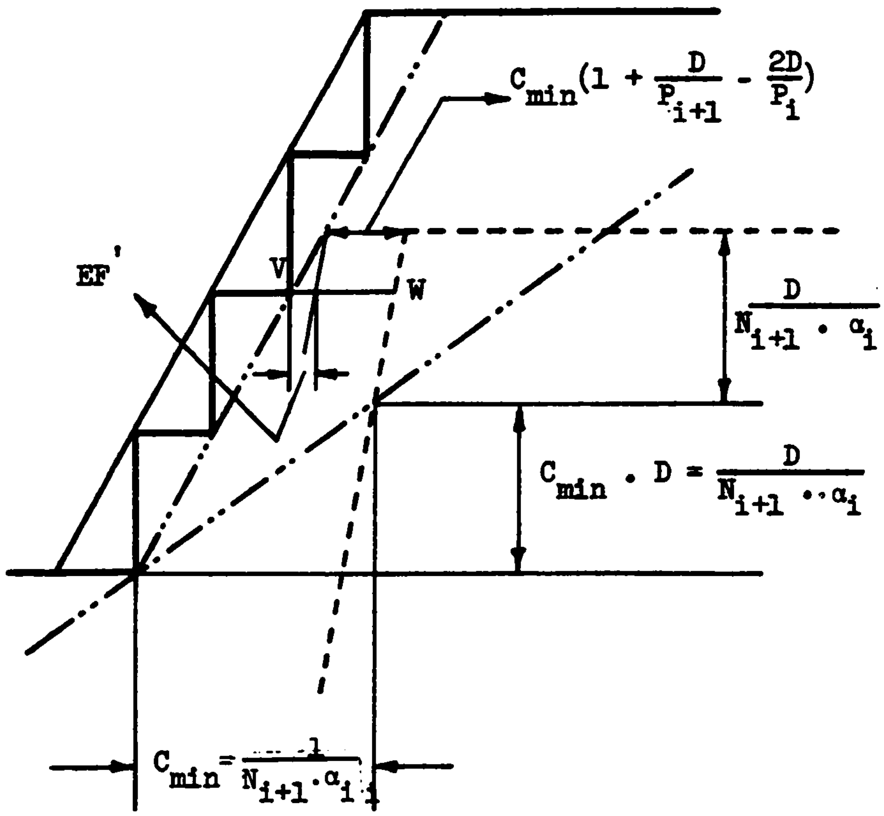

Figure A.2. A portion of Figure A.1 where the value of C_i is minimum

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V and W is $C_{min}\left(1 + \frac{D}{P_{i+1}} - \frac{2D}{P_i}\right) + CG + EF'$

$= \frac{1}{N_{i+1} \cdot \alpha_i}\left(1 - \frac{D}{P_i}\right) + EF'$. As long as this distance is bigger than or equal to EF, no crossing occurs. Otherwise, crossing occurs and the development in Section III-A-6 will not be valid. Consequently, the necessary condition for the validity of the development is

$$\frac{1}{N_{i+1} \cdot \alpha_i} \left(1 - \frac{D}{P_i}\right) + EF' \geq EF .$$

By using the relation $K_i' - K_i < \frac{n_i}{\beta_{i+1}} + \frac{1}{2}$, the above condition can be reduced to

$$n_i \geq \frac{\beta_{i+1}}{2(P_i - D)} \cdot \frac{D(P_{i+1} - P_i)}{P_{i+1}} .$$

IX. APPENDIX B: THE PROOF OF EQUATION 4.9 FOR AN ARBITRARY LEAD TIME DISTRIBUTION

In addition to the symbols defined in Section A-2-b of Chapter IV, the following symbols are defined.

$\Psi_1(x)$: State probability that on hand inventory at any time t is x $(0 \leq x \leq r + Q)$

$\Psi_2(y)$: State probability that backorders at any time t are y $(0 \leq y)$

$B(Q,r)$: Expected number of backorders at any time t

$D(Q,r)$: Expected number of on hand inventory at any time t

It is possible to express $\Psi_1(x)$ and $\Psi_2(y)$ in terms of $H(x)$ as follows:

$$\Psi_1(x) = \frac{1}{Q} \sum_{j=1}^{Q} h(r + j - x) = \frac{1}{Q} [H(r + 1 - x) - H(r + Q + 1 - x)],$$

$$(0 \leq x \leq r) \qquad (A.1)$$

$$\Psi_1(x) = \frac{1}{Q} \sum_{j=x-r}^{Q} h(r + j - x) = \frac{1}{Q} [1 - H(r + Q + 1 - x)],$$

$$(r + 1 \leq x \leq r + Q) \qquad (A.2)$$

$$\Psi_2(y) = \frac{1}{Q} \sum_{j=1}^{Q} h(r + j + y) = \frac{1}{Q} [H(r + 1 + y) - H(r + Q + y + 1)],$$

$$(0 \leq y) \qquad (A.3)$$

where

$$P_{out} = \sum_{y=0}^{\infty} \Psi_2(y) = \frac{1}{Q}\left[\sum_{y=0}^{\infty} H(r + 1 + y) - \sum_{y=0}^{\infty} H(r + Q + y + 1)\right]. \tag{A.4}$$

Then $B(Q,r)$ is

$$B(Q,r) = \sum_{y=0}^{\infty} y\Psi_2(y) = \frac{1}{Q}\sum_{y=0}^{\infty} y[H(r + 1 + y) - H(r + Q + y + 1)]$$

$$= \frac{1}{Q}\left[\sum_{w=r+1}^{\infty} (w - r - 1)\, H(w) - \sum_{w=r+Q+1}^{\infty} (w - r - Q - 1)\, H(w)\right]. \tag{A.5}$$

The average WIP in storage in unit-years, $D(Q,r)$, is

$$D(Q,r) = \sum_{x=0}^{r+Q} x\Psi_1(x) = \sum_{x=0}^{r} x\Psi_1(x) + \sum_{x=r+1}^{r+Q} x\Psi_1(x)$$

$$= \frac{1}{Q}\sum_{x=0}^{r} x[H(r + 1 - x) - H(r + Q + 1 - x)]$$

$$+ \frac{1}{Q}\sum_{x=r+1}^{r+Q} x[1 - H(r + Q + 1 - x)]$$

$$= \frac{1}{Q}\left[Qr + \frac{Q(Q+1)}{2}\right] + \frac{1}{Q}\sum_{x=0}^{r} xH(r + 1 - x) - \frac{1}{Q}\sum_{x=0}^{r+Q} xH(r + Q + 1 - x)$$

$$= r + \frac{Q+1}{2} + \frac{1}{Q}\left[\sum_{w=1}^{\infty} (r + 1 - w)\, H(w) + \sum_{w=r+1}^{\infty} (w - r - 1)\, H(w)\right.$$

$$- \sum_{w=1}^{\infty} (r+Q+1-w)\, H(w) - \sum_{w=r+Q+1}^{\infty} (w - r - Q - 1)\, H(w)]$$

$$= r + \frac{Q+1}{2} + \frac{1}{Q} [\, -Q \sum_{w=1}^{\infty} H(w)] + B(Q,r) \ .$$

Since $\sum_{w=1}^{\infty} H(w) = \sum_{w=1}^{\infty} wh(w) = \mu$,

$$D(Q,r) = r + \frac{Q+1}{2} - \mu + B(Q,r) \ .$$